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Finite element approximation of the Fokker–Planck equation for diffuse optical tomography

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ABSTRACT

In diffuse optical tomography, light transport theory is used to describe photon propagation inside turbid medium. A commonly used simplification for the radiative transport equation is the diffusion approximation due to computational feasibility. However, it is known that the diffusion approximation is not valid close to the sources and boundary and in low-scattering regions. Fokker–Planck equation describes light propagation when scattering is forward-peaked. In this article a numerical solution of the Fokker–Planck equation using finite element method is developed. Approach is validated against Monte Carlo simulation and compared with the diffusion approximation. The results show that the Fokker–Planck equation gives equal or better results than the diffusion approximation on the boundary of a homogeneous medium and in turbid medium containing a low-scattering region when scattering is forward-peaked.

1. Introduction

In diffuse optical tomography (DOT), the goal is to reconstruct the optical properties of tissues using boundary measurements of scattered near infrared light. The imaging modality has potential applications for example in detection of breast cancer, neonatal brain imaging and functional brain activation studies [1,2]. In the measurement set-up, a set of optical fibers, optodes, are attached on the boundary of the object in measurement and source positions. Near infrared light is guided into the object at one source position at a time and transmitted light is measured from all the measurement positions using light sensitive detectors. Then, this measurement process is repeated for all source positions.

The image reconstruction in DOT is a nonlinear ill-posed inverse problem. The iterative solution of this problem requires several solutions of the forward problem. Moreover, due to ill-posedness of the reconstruction problem, even small errors in the modelling can produce large errors in reconstructions. Therefore, an accurate and computationally feasible forward model is needed.

Light propagation in biological tissues is governed by the transport theory [3,4]. This leads to describing the multiple scattering phenomenon in biological tissues using the radiative transport equation (RTE). Due to computational complexity of the RTE, different approximations have been developed to ease up the computation of the forward problem. A common approximation is the P_n approximation where the solution of the RTE is expanded into series of spherical harmonics.

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The most often used model for the solution of the forward problem in DOT is the diffusion approximation (DA) which is a special case of the P_1 approximation. The DA is computationally feasible but it has limitations in accuracy; it fails to describe light propagation accurately in low-scattering regions as well as in the proximity of the light source and boundaries [1,5].

Recently there has been a growing interest in using other approximations of the RTE as a forward model. The idea of using the Fokker–Planck equation as the forward model in DOT was introduced in [6]. The Fokker–Planck equation can be used to describe light propagation accurately when scattering is strongly forward dominated [7]. This is the case in biological tissues [8,6].

The derivation of the Fokker–Planck equation for light transport can be found in [9] and for particle transport in [10]. In the derivation of the Fokker–Planck equation for light transport, the scattering probability distribution is approximated by a sum of delta function and a second order correction [9]. This approximation explains the limits of the Fokker–Planck equation. It cannot describe the light propagation accurately when the scattering is not forward dominated, but on the other hand when it is the Fokker–Planck equation offers a good model for light propagation.

There are few numerical solutions to the Fokker–Planck equation. A numerical solution for the Fokker–Planck equation using discrete ordinates method was developed for particle transport in [11]. In [12], a method for computing Green's function for the Fokker–Planck equation as an expansion in plane wave modes was developed. The plane wave modes for the Fokker–Planck equation were calculated using finite difference approximation. The DOT reconstruction of the scattering and absorption coefficients using the Fokker–Planck equation as a forward model was presented in [13]. The forward problem was solved numerically using the finite difference method and the inverse problem was solved using a transport–backtransport method developed in [14].

In this paper, a finite element solution of the Fokker–Planck equation is introduced. The finite element method (FEM) is a flexible approach when implementing different boundary conditions and handling complex geometries. It has successfully been used in numerical solution of the light transport problems [15–17]. In this paper both spatial and angular variables are discretized using the FEM when solving the Fokker–Planck equation. A similar approach has earlier been used in solution of the RTE and the radiative transfer problem of ionizing radiation [17,18]. To the authors knowledge the Fokker–Planck equation has not yet been solved using the finite element method.

In the numerical solution of the RTE, dense angular dicretization is needed in order to describe light propagation accurately in strongly scattering medium [13]. Therefore, large computational resources are needed in the solutions of the RTE. The Fokker–Planck equation, on the other hand, assumes that the scattering is forward peaked. Thus, coarser angular discretization can be used in the numerical computation compared to the RTE, leading to smaller amount of computation load and time [13].

The rest of the paper is organized as follows. In Section 2, we give a short review of the RTE, the DA, and the Fokker–Planck equation. In Section 3, we derive a finite element solution for the Fokker–Planck equation. In Section 4, we test the proposed FE-model with simulations. In Section 5, conclusions are given.

2. Light transport models

Let $\Omega \subset \mathbb{R}^n$ be the physical domain and n=2,3 be the dimension of the domain. In addition, let $\hat{s} \in S^{n-1}$ denote a unit vector in the direction of interest.

The frequency domain version of the RTE is of the form

$$\frac{\mathrm{i}\omega}{c}\phi(r,\hat{s}) + \hat{s}\cdot\nabla\phi(r,\hat{s}) + \mu_a\phi(r,\hat{s}) = \mu_s L\phi(r,\hat{s}) + q(r,\hat{s}), \tag{1}$$

where i is the imaginary unit, c is the speed of light in medium, ω is the angular modulation frequency of the input signal, $q(r,\hat{s})$ is the source inside Ω, $\phi(r,\hat{s})$ is the radiance, $\mu_s = \mu_s(r)$ and $\mu_a = \mu_a(r)$ are the scattering and absorption parameters of the medium and L is the scattering operator defined as

$$L\phi(r,\hat{s}) = -\phi(r,\hat{s}) + \int_{S^{n-1}} \Theta(\hat{s},\hat{s}')\phi(r,\hat{s}')\,\mathrm{d}\hat{s}'. \tag{2}$$

The scattering phase function $\Theta(\hat{s},\hat{s}')$ describes the probability density for a photon to scatter from direction $\hat{s}'$ to direction $\hat{s}$. In this study the domain is assumed to be isotropic in the sense that probability of scattering depends only on the relative angle, not on the absolute angles, i.e. $\Theta(\hat{s},\hat{s}') = \Theta(\hat{s}\cdot\hat{s}')$. An often used phase function for isotropic materia is the Henyey–Greenstein scattering function [19] which is of the form

$$\Theta(\hat{s}\cdot\hat{s}') = \begin{cases} \dfrac{1}{2\pi}\dfrac{1-g^2}{(1+g^2-2g\hat{s}\cdot\hat{s}')} & n=2, \\ \dfrac{1}{4\pi}\dfrac{1-g^2}{(1+g^2-2g\hat{s}\cdot\hat{s}')^{3/2}} & n=3, \end{cases} \tag{3}$$

where parameter g defines the shape of the probability distribution. Values of g are in the range from -1 to 1, for $g<0$ scattering is backward dominated and for $g>0$ scattering is forward dominated. When g=0, scattering phase function is a uniform distribution. In biological tissues, g is typically close to 1.

In DOT, a natural boundary condition for the RTE is the so-called vacuum boundary condition which assumes that no photons travel in an inward direction at the boundary $\partial\Omega$, thus

$$\phi(r,\hat{s}) = 0, \quad r\in\partial\Omega, \quad \hat{s}\cdot\hat{n} < 0, \tag{4}$$

where $\hat{n}$ denotes the outward unit normal on $\partial\Omega$ [1]. The vacuum boundary condition can be modified to take into account a boundary source $\phi_0(r,\hat{s})$ at source position $\varepsilon_i \subset \partial\Omega$,

$$\phi(r,\hat{s}) = \begin{cases} \phi_0(r,\hat{s}), & r\in\bigcup_j\varepsilon_j, \quad \hat{s}\cdot\hat{n}<0, \\ 0, & r\in\partial\Omega\backslash\bigcup_j\varepsilon_j, \quad \hat{s}\cdot\hat{n}<0. \end{cases} \tag{5}$$

Photon density is defined as an integral of the radiance over angular directions

$$\Phi(r) = \int_{S^{n-1}} \phi(r,\hat{s})\,\mathrm{d}\hat{s}. \tag{6}$$