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Note

On the far field in the Lorenz–Mie theory and T-matrix formulation

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ABSTRACT

The far field within the context of the Lorenz–Mie theory and the T-matrix formulation is usually expressed on the basis of the asymptotic properties of vector spherical waves. The radiation condition is taken into account by employing proper vector spherical functions as the expansion basis of the scattered field. The asymptotic behavior of the Hankel function is obtained from differential equations. The asymptotic far field can also be obtained from the Kirchhoff surface integral equation, in which the radiation condition has been implemented when it is derived from the Maxwell equations. This note is to present an explicit establishment of the relationship between the asymptotic far field and the near field in the Lorenz–Mie theory and the T-matrix formulation through the Kirchhoff surface integral.

1. Introduction

The Lorenz–Mie theory [1] and the T-matrix [2] formulation provide exact solutions for the scattering of electromagnetic waves by small particles. It has been demonstrated that the Lorenz–Mie theory is a special case of the T-matrix method when the latter is applied to spheres [3]. In the two methods, the incident, scattered and internal fields are expanded in terms of vector spherical wave functions. A recent paper by Mishchenko [4] discussed the fundamental concepts of electromagnetic scattering. The expansion coefficients in the Lorenz–Mie theory and the T-matrix formulation are determined from the boundary condition and extended boundary condition (EBC), respectively. Two aspects associated with the T-matrix formulation should be addressed. First, the method to calculate the T-matrix is not restricted to EBC as other methods can be employed [5–7]. Second, other expansion bases (e.g., vector spheroidal/ellipsoidal wave functions) can be employed to expand the electromagnetic fields [8,9]. In this paper, the scattered field is written in the form of

$$\vec{E}^s(\vec{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[p_{mn}\vec{M}_{mn}(k\vec{r}) + q_{mn}\vec{N}_{mn}(k\vec{r})], \tag{1}$$

where $\vec{M}_{mn}$ and $\vec{N}_{mn}$ are the vector spherical wave functions, which are transverse at infinity [3,10], k is the wave number, and p_{mn} and q_{mn} are the expansion coefficients. The vector spherical wave functions are related to vector spherical harmonics [3] given by

$$\vec{M}_{mn}(kr,\theta,\varphi) = \gamma_{mn}h_n^{(1)}(kr)\vec{C}_{mn}(\theta,\phi), \tag{2}$$

$$\vec{N}_{mn}(kr,\theta,\varphi) = \gamma_{mn}\left\{\frac{n(n+1)}{kr}h_n^{(1)}(kr)\vec{P}_{mn}(\theta,\phi) + \frac{1}{kr}\frac{d}{d(kr)}(krh_n^{(1)}(kr))\vec{B}_{mn}(\theta,\phi)\right\}, \tag{3}$$

where $h_n^{(1)}(kr)$ is the Hankel function of the first kind, $\vec{C}_{mn}$, $\vec{B}_{mn}$, and $\vec{P}_{mn}$ are vector spherical harmonics and γ_{mn} is a defined constant and equal to $\sqrt{(2n-1)(n-m)/4\pi(n+1)(n+m)}$. The solution in the radiation zone can be expressed via the asymptotic forms of $\vec{M}_{mn}$ and $\vec{N}_{mn}$ [3] as follows:

$$\vec{M}_{mn}(kr,\theta,\varphi) = \frac{(-i)^{n+1}e^{ikr}}{kr}\gamma_{mn}\vec{C}_{mn}(\theta,\phi), \tag{4}$$

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$$\vec{N}_{mn}(kr,\theta,\varphi)=\frac{(-i)^n e^{ikr}}{kr}\gamma_{mn}\vec{B}_{mn}(\theta,\phi). \tag{5}$$

Note that the expansion of the scattered field in terms of $\vec{M}_{mn}$ and $\vec{N}_{mn}$ in Eq. (1) is due to the above asymptotic behaviors, and has taken into account the radiation condition. The preceding asymptotic properties are obtained by analyzing the differential equation satisfied by the Hankel function. Specifically, the scattered far field is given by

$$\vec{E}^s(\vec{r})=\frac{e^{ikr}}{-ikr}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}i^{-(n+1)}\gamma_{mn}[-ip_{mn}\vec{C}_{mn}(\theta^s,\phi^s)+q_{mn}\vec{B}_{mn}(\theta^s,\phi^s)], \tag{6}$$

where θ^s and ϕ^s are the polar zenith angle and azimuthal angle of the scattering direction, respectively.

The far field can also be formulated in terms of the so-called Kirchhoff surface integral [11]:

$$\vec{E}^s(\vec{r})=\frac{e^{ikr}}{-ikr}\frac{k^2}{4\pi}\int\left\{\hat{r}\times[\hat{n}_s\times\vec{E}(\vec{r}\,')]-\hat{r}\times\hat{r}\times[\hat{n}_s\times\vec{H}(\vec{r}\,')]\right\}e^{-ik\hat{r}\cdot\vec{r}\,'}d^2\vec{r}\,', \tag{7}$$

where $\hat{r}$ is the unit vector pointing to the observation position, $\vec{E}$ and $\vec{H}$ are electric and magnetic fields on an arbitrary surface enclosing the particle, and $\hat{n}_s$ is the outward normal direction at the position $\vec{r}\,'$. When the refractive index of the particle is equal to unity, $\vec{E}$ and $\vec{H}$ are equal to the incident electromagnetic fields and $\vec{E}^s$ is zero. Therefore, $\vec{E}$ and $\vec{H}$ in Eq. (7) can be either the scattered field or the total field. In this method, the radiation condition has been incorporated into the integral (7), when it is derived from the Maxwell equations. The Huygens principle is explicit in this formulation.

Fundamentally, the scattered field expressed in formula (1) should satisfy the integral in Eq. (7). In appearance, the near field and far field are radially correlated. If we substitute Eq. (1) and the associated magnetic field into Eq. (7), the same solution given by Eq. (6) can be obtained. This process has been performed numerically for the scattering by spheres [12] to examine the accuracy the implementation of near-to-far field transformation in the finite-difference time-domain (FDTD) method. In the following discussion, we will show that the same asymptotic formulation can be analytically obtained from Eq. (7). To the best of our knowledge, although this relation is implied, it is not explicitly proven in the literature. For convenience, we rewrite Eq. (7) as two equations:

$$\hat{\alpha}^s\cdot\vec{E}^s(\vec{r})=\frac{e^{ikr}}{-ikr}\frac{k^2}{4\pi}\int\left\{\hat{\beta}^s\cdot[\hat{n}_s\times\vec{E}(\vec{r}\,')]+\hat{\alpha}^s\cdot[\hat{n}_s\times\vec{H}(\vec{r}\,')]\right\}e^{-ik\hat{r}\cdot\vec{r}\,'}d^2\vec{r}\,' \tag{8}$$

$$\hat{\beta}^s\cdot\vec{E}^s(\vec{r})=\frac{e^{ikr}}{-ikr}\frac{k^2}{4\pi}\int\left\{-\hat{\alpha}^s\cdot[\hat{n}_s\times\vec{E}(\vec{r}\,')]+\hat{\beta}^s\cdot[\hat{n}_s\times\vec{H}(\vec{r}\,')]\right\}e^{-ik\hat{r}\cdot\vec{r}\,'}d^2\vec{r}\,', \tag{9}$$

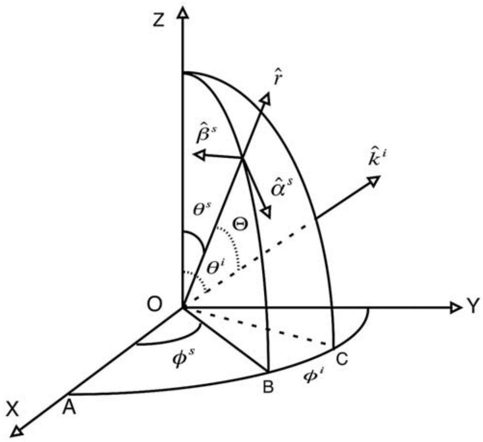

Fig. 1. Illustration of the incident direction $\hat{k}^i$, scattered direction $\hat{r}$ and associated unit vectors. Θ is the scattering angle. When the incident plane wave is along the z axis, θ^s is equal to the scattering angle.

where $\hat{\alpha}^s$ and $\hat{\beta}^s$ are two unit vectors parallel and perpendicular to the plane defined by the z axis and the scattering direction, as illustrated in Fig. 1. In a spherical coordinate system, we have

$$\hat{\alpha}^s=\hat{\theta}^s,\hat{\beta}^s=-\hat{\phi}^s. \tag{10}$$

If the incident field is along the z axis, this plane is called the scattering plane and $\hat{\theta}^s$ is the scattering angle.

2. Expansion of polarized plane wave

To analytically integrate the integrals in Eqs. (8) and (9), it is important to expand the polarized plane waves $\hat{\beta}^s e^{-ik\hat{r}\cdot\vec{r}\,'}$ and $\hat{\alpha}^s e^{-ik\hat{r}\cdot\vec{r}\,'}$ in terms of vector spherical wave functions as follows:

$$\hat{\beta}^s e^{-ik\hat{r}\cdot\vec{r}\,'}=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}[a_{\beta,mn}\mathrm{Rg}^*\vec{M}_{mn}(kr,\theta,\phi)+b_{\beta,mn}\mathrm{Rg}^*\vec{N}_{mn}(kr,\theta,\phi)], \tag{11}$$

$$\hat{\alpha}^s e^{-ik\hat{r}\cdot\vec{r}\,'}=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}[a_{\alpha,mn}\mathrm{Rg}^*\vec{M}_{mn}(kr,\theta,\phi)+b_{\alpha,mn}\mathrm{Rg}^*\vec{N}_{mn}(kr,\theta,\phi)], \tag{12}$$

where $\mathrm{Rg}^*\vec{M}_{mn}$ and $\mathrm{Rg}^*\vec{N}_{mn}$ are the conjugates of the regular vector spherical functions, which are defined by replacing the Hankel function in Eqs. (2) and (3) by the spherical Bessel function. To determine the coefficients, we consider $a_{\beta,mn}$ as an example, given by

$$\begin{aligned}a_{\beta,mn}&=\frac{\gamma_{mn}}{j_n(kr)}\int\hat{\beta}^s e^{-ik\hat{r}\cdot\vec{r}}\cdot\vec{C}_{mn}(\theta,\phi)\sin\theta\,d\theta\,d\phi\\&=4\pi(-i)^n\gamma_{mn}\hat{\beta}^s\cdot\vec{C}_{mn}(\theta^s,\phi^s)\\&=4\pi(-i)^n\gamma_{mn}\tau_{mn}(\theta^s)(-1)^m\sqrt{\frac{(n+m)!}{(n-m)!}}e^{im\phi^s}.\end{aligned} \tag{13}$$