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On vector radiative transfer equation in curvilinear coordinate systems

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ABSTRACT

The differential operator of polarized radiative transfer equation is examined in case of homogeneous medium in Euclidean three-dimensional space with arbitrary curvilinear coordinate system defined in it. This study shows that an apparent rotation of polarization plane along the light ray with respect to the chosen reference plane for Stokes parameters generally takes place, due to purely geometric reasons. Analytic expressions for the differential operator of transfer equation dependent on the components of metric tensor and their derivatives are found, and the derivation of differential operator of polarized radiative transfer equation has been made a standard procedure. Considerable simplifications take place if the coordinate system is orthogonal.

1. Introduction

Transfer equation for polarized radiation has a long history of development. One of its early versions was that by Chandrasekhar [1] for Rayleigh scattering in plane–parallel layer. Not attempting to make any substantial historical review of the further development nor to study scientific priority ownerships I shall briefly refer to some of the most important forms of the transfer equation available in the literature.

Kuščer and Ribarič [2] seem to be the first who split the scattering kernel for polarized radiation within isotropic medium into generalized spherical functions. Transfer equation in case of spatially variable real part of the refraction index and consequently curvilinear ray paths can be found in [3,4]. Transfer equation in anisotropic magnetized plasma with birefringence was widely reviewed in [5], while polarized radiative transfer including Compton effect was considered in [6] and elsewhere.

Along with continuously growing number of observations of such astronomical objects where general relativity is essential, theoretical consideration of radiation transfer in curved spacetime has been steadily developing—see, for example, [7–11]. It was shown in [10] that while the point of observation moves along the light ray, the linear polarization plane seemingly rotates in the vacuum due to the rotation of the polarization reference plane (one can call it Stokes reference plane) around the direction of propagation.

Another case of rotation of polarization plane due to essentially geometric reasons was remarked in [12, Paragraph 85], namely, this effect occurs if the electromagnetic radiation in medium with inhomogeneous refraction index moves along a curve with nonzero torsion (in terms of geometrical optics approximation). A quite fundamental investigation of the rotation of linear polarization plane as well as conversions between linear and elliptic polarization during the propagation of radiation in inhomogeneous and/or anisotropic medium was done in [13].

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In fact, similar effects can be observed without any gravitation field, in homogeneous isotropic medium. This paper shows that very often the true rotation of Stokes reference plane and the corresponding apparent rotation of linear polarization plane around the direction of propagation in vacuum takes place in Minkowski (i.e. flat) spacetime and consequently Euclidean space, with spatially constant real part of the effective refractive index (determined by both the refractive index of the host medium and by extinction on sparsely placed polydisperse scattering particles), on condition that the space is considered within the framework of some curvilinear coordinate system. The simplest example is cylindrical coordinate system with radial spatial direction taken as both the polar axis for the determination of the direction of propagation of radiation and simultaneously as one of the two directions defining Stokes reference plane. This certainly is not a physical rotation of polarization plane. The observable effect arises from the fact that the Stokes reference plane, as a rule, is tied to the spatial coordinate system and its basis vectors.

This effect requires a generalized consideration. While modeling some object, the spatial coordinate system should reflect the symmetry of the physical problem, and there are physical objects of many different forms in the nature. One can mention bipolar (approximately toroidal, conical and ellipsoidal) dust–gas nebulosities around late-type stars and protoplanetary nebulae in our Galaxy—see, e.g. pictures in [14–16]. Undoubtedly, the mentioned astronomical objects are not perfectly symmetric, and the symmetry of the radiation field typically is lower than the symmetry of the distribution of matter. This is also true when modeling not perfectly plane–parallel nor perfectly spherical stellar and planetary atmospheres (with active regions, sunspots, clouds, etc.) in plane–parallel or spherical approximation.

The object of this study is the differential operator of vector radiative transfer equation because this operator is dependent on the chosen coordinate system. The other terms (extinction, scattering and primary sources) are essentially the same in all coordinate systems. It is stressed that the ladder specific coherence dyadic [17,18] is a tensor, and its spatial derivative should be taken as the absolute derivative [19,20]. Tensor analysis proves to be useful in order to obtain full expression for the differential operator of vector radiative transfer equation in arbitrary curvilinear coordinate system by means of unified standardized routine calculation—similarly as to calculate divergence or curl of some vector in some coordinate system is a routine task, without necessity to make any tricks of differential geometry.

Section 2 introduces the designations for general spatial coordinate system, defines the directional angles of ray propagation, and presents calculations of the spatial derivatives of directional angles along light ray. Section 3 reviews the tensor properties of dyadics used in the theory of electromagnetic scattering [17,18]. The correct form of vector integro-differential radiative transfer equation in arbitrary curvilinear coordinate system is derived in Section 4. Section 5 briefly summarizes the main results. Section 6 specifies the role of funding sources and the limits of their influence on this paper.

2. Spatial and angular coordinates

In this paper vector, tensor and linear connection indices are used as follows: if the letters i, j, k, l, … are written in some product as both lower (covariant) and upper (contravariant) indices, for example, in expressions of type $E_i E^i$, $\Gamma^i_{kl} E^k \Omega^l$, Γ^i_{ki}, then the summation over the repeated indices within the limits of dimensionality of space is assumed, as it is commonly adopted in tensor calculus [19]. But if the first three letters of Latin alphabet (a, b, c) are used the same way (for example, $\Gamma^a_{ba} E^b$), no summation occurs; a, b and c are assumed to be fixed (this fixation will be done a little bit below).

Let us consider three-dimensional Euclidean space with some coordinate system x^i, $i=1, 2, 3$, defined in it. Let γ_{ik} be the covariant components of the metric tensor [19–22], i.e. square of length of the arc element is

$$ds^2 = \gamma_{ik}\, dx^i\, dx^k \tag{1}$$

(following [21], I use the designation γ_{ik} for the metric tensor of three-dimensional space, in order to preserve the more usual designation g_{ik} for the metric tensor of four-dimensional spacetime). Let us assume that in almost all space the metric tensor has a finite determinant

$$\Gamma \overset{\text{def}}{\equiv} \det \gamma_{ik} > 0, \tag{2}$$

with possible exception on a finite set of surfaces and lines (with three-dimensional Lebesgue's measure equal to zero), where Γ can be either unbounded or zero.

The contravariant components of the vector $\mathbf{\Omega}$ defining the direction of propagation of radiation can be defined as

$$\Omega^i = \frac{dx^i}{ds}, \tag{3}$$

where the derivative of spatial coordinate x^i by path s while moving along the light ray is taken. According to [19] and [22, Table 6.3-1], in general nonorthogonal coordinate system one can write either

$$\mathbf{\Omega} = \Omega^i \mathbf{e}_i \tag{4}$$

or

$$\mathbf{\Omega} = \Omega_i \mathbf{e}^i, \tag{5}$$

where

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i} \tag{6}$$

is the covariant basis vector tangent to the coordinate line of increasing x^i, $\mathbf{r}$ being the radius vector,

$$\Omega_i = \gamma_{ik} \Omega^k \tag{7}$$

are the covariant components of the vector $\mathbf{\Omega}$, and

$$\mathbf{e}^i = \gamma^{ik} \mathbf{e}_k \tag{8}$$

are the contravariant basis vectors. The contravariant components of the metric tensor γ^{ij} are defined by

$$\gamma^{ij} \gamma_{jk} = \delta^i_k, \tag{9}$$