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A probabilistic study of the influence of parameter uncertainty on solutions of the radiative transfer equation

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ABSTRACT

The influence of uncertainty in the absorption and scattering coefficients on the solution and associated parameters of the radiative transfer equation is studied using polynomial chaos theory. The uncertainty is defined by means of uniform and log-uniform probability distributions. By expanding the radiation intensity in a series of polynomial chaos functions we may reduce the stochastic transfer equation to a set of coupled deterministic equations, analogous to those that arise in multigroup neutron transport theory, with the effective multigroup transfer scattering coefficients containing information about the uncertainty. This procedure enables existing transport theory computer codes to be used, with little modification, to solve the problem. Applications are made to a transmission problem and a constant source problem in a slab. In addition, we also study the rod model for which exact analytical solutions are readily available. In all cases, numerical results in the form of mean, variance and sensitivity are given that illustrate how absorption and scattering coefficient uncertainty influences the solution of the radiative transfer equation.

1. Introduction

An important source of error in calculations employing the radiative transfer (RT) equation arises from uncertain data values. These errors originate from lack of knowledge about the absorption and scattering coefficients (parameters). It is vital, therefore, to know how changes in these parameters affect the outcome of our calculations. For example, what are the uncertainties in the reflection and transmission coefficients in relation to uncertainties in the absorption and scattering coefficients? When the uncertainties are relatively small, these matters can be been handled by perturbation theory [1,2]. However, there is another effective method of dealing with such a problem which does not have the limitations of perturbation theory and which may be used for arbitrarily large parameter uncertainties; namely, polynomial chaos expansions. In order to proceed, therefore, we deliberately introduce uncertainty into the parameters of the transport equation. This is done by assuming that the parameters are functions of a random variable which, in turn, allows us to prescribe a probability distribution for the values of the parameters. The polynomial chaos expansion (PCE) [3–8] will be employed to transform the stochastic RT equation into a set of coupled deterministic transport equations for the coefficients in the PCE. By a further transformation, these coupled equations may be cast into a form that is analogous to a multigroup neutron or radiation transport formulation but with probabilistic matrix elements taking the place of the group-to-group energy exchange cross sections. The only difference between the equations derived below from those of multigroup theory is the presence of coupled boundary

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conditions. This transformation therefore enables us to use well-established computer codes to solve the associated RT equation.

In this paper we will give examples based upon the one-dimensional RT equation for a slab. Two cases are considered: (1) a constant source in a bare slab and (2) the transmission of radiation through a plane slab with an isotropic source incident on one face. We will solve the RT equation numerically but because certain analytic solutions exist for the cases we have chosen, we will be able to compare the PCE method with the exact result for mean and variance and hence estimate convergence rates. We also solve the problems using the rod model, for which exact solutions are also available over the complete range of distance. The calculations, which use the exact radiative transfer equation, are carried out by a modification of the computer code EVENT [9]. The general conclusion is that the PCE method shows considerable promise and can deal with a variety of probability distribution functions describing parameter uncertainty.

It is important to note that a number of other fields of technology have benefited from the use of PCE to study the effects of data uncertainty. For example, the uncertainty of modes of vibration in structural mechanics due to errors in elastic constants [10], the flow of groundwater in the geosphere due to rock fractures [11] and uncertainties in fluid flow due to both geometric uncertainty and basic data [12]. The extension to radiative transfer is an obvious extension as we hope to convince the reader.

2. Theory

The radiative transfer (RT) equation for the radiation intensity $I(\mathbf{r},\mathbf{\Omega},v;\xi)$, may be written as [13]

$$\begin{aligned}\mathbf{\Omega}.\nabla I(\mathbf{r},\mathbf{\Omega},v;\xi)+\kappa(v;\xi)I(\mathbf{r},\mathbf{\Omega},v;\xi)\\ =\int dv'\int d\mathbf{\Omega}'\kappa_s(v';\xi)f(v'\to v,\mathbf{\Omega}.\mathbf{\Omega}';\xi)I(\mathbf{r},\mathbf{\Omega}',v';\xi)\\ +S(\mathbf{r},\mathbf{\Omega},v;\xi).\end{aligned} \tag{1}$$

In Eq. (1) the extinction coefficient $\kappa(v;\xi)=\kappa_a(v;\xi)+\kappa_s(v;\xi)$, where κ_a and κ_s are the absorption and scattering coefficients at frequency v, respectively. $f(v'\to v,\mathbf{\Omega}.\mathbf{\Omega}';\xi)$ is the energy and angle exchange phase function. $S(\mathbf{r},\mathbf{\Omega},v;\xi)$ is a source term which itself may contain uncertainties and is the Planck function and/or an independent source of photons. The $\xi(\xi_1,\xi_2,\ldots,\xi_N)$ are independent random variables which will describe the data uncertainty.

Let us assume that the random intensity can be expanded in a PCE in the form [3,14]

$$I(\mathbf{r},\mathbf{\Omega},v;\xi)=\sum_{n=0}^{N}I_n(\mathbf{r},\mathbf{\Omega},v)\Phi_n(\xi) \tag{2}$$

where $\Phi_n(\xi)$ are the polynomial chaos functions which are to be determined by the nature of the problem and obey the orthogonality relation

$$\langle\Phi_n(\xi)\Phi_m(\xi)\rangle\equiv\int_R d\xi p(\xi)\Phi_n(\xi)\Phi_m(\xi)=\delta_{nm}N_m^2 \tag{3}$$

$p(\xi)$ being an appropriate probability weight function of the random variables ξ. Eq. (3) defines the normalisation coefficient N_m. For full details of the concept of polynomial chaos functions, the reader is referred to Ghanem and Spanos [3]. However, in the simplest case of one random variable, ξ, the polynomials form a complete set of orthogonal functions with the weight function given by the appropriate probability distribution [15]. For more than one random variable, the sequence of polynomials can be generalised as explained in [3] or [14].

The expansion coefficients $I_n(\mathbf{r},\mathbf{\Omega},v)$ are to be determined as we will show below. Inserting Eq. (2) into Eq. (1) we find

$$\begin{aligned}\mathbf{\Omega}.\nabla\sum_{n=0}^{N}I_n(\mathbf{r},\mathbf{\Omega},v)\Phi_n(\xi)+\kappa(v;\xi)\sum_{n=0}^{N}I_n(\mathbf{r},\mathbf{\Omega},v)\Phi_n(\xi)\\ =\int dv'\int d\mathbf{\Omega}'\kappa_s(v';\xi)f(v'\to v,\mathbf{\Omega}.\mathbf{\Omega}';\xi)\\ \sum_{n=0}^{N}I_n(\mathbf{r},\mathbf{\Omega}',v')\Phi_n(\xi)+S(\mathbf{r},\mathbf{\Omega},v;\xi).\end{aligned} \tag{4}$$

Now multiplying Eq. (3) by $p(\xi)\Phi_m(\xi)$ and averaging over all the random variables ξ, leads to

$$\begin{aligned}\mathbf{\Omega}.\nabla I_m(\mathbf{r},\mathbf{\Omega},v)\langle\Phi_m^2(\xi)\rangle+\sum_{n=0}^{N}I_n(\mathbf{r},\mathbf{\Omega},v)\langle\Phi_m(\xi)\kappa(v;\xi)\Phi_n(\xi)\rangle\\ =\langle\Phi_m(\xi)S(\mathbf{r},\mathbf{\Omega},v;\xi)\rangle+\int dv'\int d\mathbf{\Omega}'\sum_{n=0}^{N}I_n(\mathbf{r},\mathbf{\Omega}',v')\\ \langle\Phi_m(\xi)\kappa_s(v';\xi)f(v'\to v,\mathbf{\Omega}.\mathbf{\Omega}';\xi)\Phi_n(\xi)\rangle.\end{aligned} \tag{5}$$

Let us now define

$$A_{mn}=\frac{\langle\Phi_m(\xi)\kappa(v;\xi)\Phi_n(\xi)\rangle}{N_nN_m} \tag{6}$$

$$B_{mn}(v'\to v,\mathbf{\Omega}.\mathbf{\Omega}')=\frac{\langle\Phi_m(\xi)\kappa_s(v;\xi)f(v'\to v,\mathbf{\Omega}.\mathbf{\Omega}';\xi)\Phi_n(\xi)\rangle}{N_nN_m} \tag{7}$$

$$S_m(\mathbf{r},\mathbf{\Omega},v)=\langle\Phi_m(\xi)S(\mathbf{r},\mathbf{\Omega},v;\xi)\rangle/N_m.$$

We also define the modified intensity polynomial chaos expansion coefficient $\hat{\phi}_m$ as

$$\hat{\phi}_m(\mathbf{r},\mathbf{\Omega},v)=N_mI_m(\mathbf{r},\mathbf{\Omega},v). \tag{8}$$

We observe that the matrices A and B are symmetric. The angular brackets in Eqs. (4)–(7) are defined by

$$\langle\Phi_m\kappa\Phi_n\rangle=\int d\xi p(\xi)\kappa(v;\xi)\Phi_m(\xi)\Phi_n(\xi) \tag{9}$$

where $p(\xi)$ is the weight function associated with PCE polynomials $\Phi_n(\xi)$. It should be noted that the evaluation of this matrix element depends on the nature of the statistics of $\kappa(v;\xi)$ and ξ. For example, suppose that κ obeys a prescribed probability distribution $p_1(\kappa)$ and suppose that we have decided that the weight function associated with the PCE polynomial $\Phi_n(\xi)$ is $p(\xi)$; if these two distributions span different spaces then the integral (9) as it stands needs interpretation. If we consider the case where there is only one random variable ξ then we can define a uniformly distributed random variable $u\in(0,1)$, and write [16]

$$u=\int_{\kappa_0}^{\kappa}p_1(\kappa')\,d\kappa'. \tag{10}$$