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Length-Two Representations of Quantum Affine Superalgebras and Baxter Operators

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Abstract: Associated to quantum affine general linear Lie superalgebras are two families of short exact sequences of representations whose first and third terms are irreducible: the Baxter TQ relations involving infinite-dimensional representations; the extended T-systems of Kirillov–Reshetikhin modules. We make use of these representations over the *full* quantum affine superalgebra to define Baxter operators as transfer matrices for the quantum integrable model and to deduce Bethe Ansatz Equations, under genericity conditions.

Contents

Introduction

Fix $\mathfrak{g} := \mathfrak{gl}(M|N)$ a general linear Lie superalgebra and q a non-zero complex number that is not a root of unity. Let $U_q(\widehat{\mathfrak{g}})$ be the associated quantum affine superalgebra [48]. This is a Hopf superalgebra neither commutative nor co-commutative, and it can be seen as a q-deformation of the universal enveloping algebra of the affine Lie superalgebra of central charge zero $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

In this paper we study a tensor category of (finite- and infinite-dimensional) representations of $U_q(\widehat{\mathfrak{g}})$. Its Grothendieck ring turns out to be commutative as is common in Lie Theory. We produce various identities of isomorphism classes of representations, and interpret them as functional relations of transfer matrices in the quantum integrable system attached to $U_q(\widehat{\mathfrak{g}})$, the XXZ spin chain.

1. Baxter operators. In an exactly solvable model a common problem is to find the spectrum of a family $T(z)$ of commuting endomorphisms of a vector space V depending on a complex spectral parameter z, called transfer matrices. The Bethe Ansatz method, initiated by H. Bethe, gives explicit eigenvectors and eigenfunctions of $T(z)$ in terms of solutions to a system of algebraic equations, the Bethe Ansatz equations (BAE). Typical examples are the Heisenberg spin chain and the ice model.

In [2], for the 6-vertex model R. Baxter related $T(z)$ to another family of commuting endomorphisms $Q(z)$ on V by the relation:

$$\text{TQ relation:} \qquad T(z) = a(z)\frac{Q(zq^2)}{Q(z)} + d(z)\frac{Q(zq^{-2})}{Q(z)}.$$

Here $a(z), d(z)$ are scalar functions and q is the parameter of the model. $Q(z)$ is a polynomial in z, called the Baxter operator. The cancellation of poles at the right-hand side becomes Bethe Ansatz equations for the roots of $Q(z)$. A similar operator equation holds for the 8-vertex model [2], where the Bethe Ansatz method fails.

Within the framework of Quantum Inverse Scattering Method, the transfer matrix $T(z)$ is defined in terms of representations of a quantum group $\mathbf{U}$. Let $\mathcal{R}(z) \in \mathbf{U}^{\otimes 2}$ be the universal R-matrix with spectral parameter z and let V, W be two representations of $\mathbf{U}$. Then $t_W(z) := \mathrm{tr}_W(\mathcal{R}(z)_{W\otimes V})$ forms a commuting family of endomorphisms on V, thanks to the quasi-triangularity of $(\mathbf{U}, \mathcal{R}(z))$. As examples, the transfer matrix for the 6-vertex model (resp. XXX spin chain) comes from tensor products of two-dimensional irreducible representations of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ (resp. Yangian $Y_\hbar(\mathfrak{sl}_2)$), while the face-type model of Andrews–Baxter–Forrester, which is equivalent to the 8-vertex model by a vertex-IRF correspondence, requires Felder's elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ [20,21].

The representation meaning of the $Q(z)$ was understood in the pioneer work of Bazhanov–Lukyanov–Zamolodchikov [3] for $U_q(\widehat{\mathfrak{sl}_2})$, and extended to an arbitrary non-twisted affine quantum group $U_q(\widehat{\mathfrak{a}})$ of a finite-dimensional simple Lie algebra $\mathfrak{a}$ in the recent work of Frenkel–Hernandez [24]. One observes that the first tensor factor of $\mathcal{R}(z)$ lies in a Borel subalgebra $U_q(\mathfrak{b})$ of $U_q(\widehat{\mathfrak{a}})$, so the above transfer-matrix construction makes sense for $U_q(\mathfrak{b})$-modules. Notably the Baxter operators $Q(z)$ are transfer matrices of $L_{i,a}^+$, the *positive prefundamental modules* over $U_q(\mathfrak{b})$, for i a Dynkin node of $\mathfrak{a}$ and $a \in \mathbb{C}^\times$. The $L_{i,a}^+$ are irreducible objects of a category $\mathcal{O}_{\mathrm{HJ}}$ of $U_q(\mathfrak{b})$-modules introduced by Hernandez–Jimbo [34].

Making use of the prefundamental modules, Frenkel–Hernandez [24] solved a conjecture of Frenkel–Reshetikhin [27] on the spectra of the quantum integrable system, which connects eigenvalues of transfer matrices $t_W(z)$, for W finite-dimensional $U_q(\widehat{\mathfrak{a}})$-modules, with polynomials arising as eigenvalues of the Baxter operators.

The two-term TQ relations, as a tool to derive Bethe Ansatz Equations for the roots of Baxter polynomials, are consequences of identities in the Grothendieck ring $K_0(\mathcal{O}_{\mathrm{HJ}})$ of category $\mathcal{O}_{\mathrm{HJ}}$ [18,19,24,25,35]. Such identities are also examples of cluster mutations of Fomin–Zelevinsky [35].