Acta Materialia 123 (2017) 245–254

Contents lists available at ScienceDirect

Acta Materialia

journal homepage: www.elsevier.com/locate/actamat

Full length article

Structural metamaterials with Saint-Venant edge effect reversal

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A R T I C L E I N F O

Article history:
Received 9 August 2016
Received in revised form
17 October 2016
Accepted 18 October 2016
Available online 29 October 2016

Keywords:
Mechanical metamaterial
Saint-Venant edge effect
Fourier methods

A B S T R A C T

When a usual material is loaded statically at surfaces, fine fluctuations of surface strain diminish fast in the material volume with the distance to the surface, a phenomenon widely known as the Saint-Venant edge effect. In this paper, highly nonlocal discrete lattices are explored to demonstrate structural metamaterials featuring *reversal* of the Saint-Venant edge effect. In these materials, certain coarse patterns of surface strain may decay faster than the finer ones. This phenomenon is shown to arise from anomalous behavior of the Fourier modes of static deformation in the material, and creates opportunities for blockage, qualitative modification and in-situ recognition of surface load patterns. Potential applications and useful practical techniques of spectral analysis of deformation, density of states and phase diagram mapping are outlined.

1. Introduction

The notion of metamaterials refers to an exciting class of man-made material systems with engineered internal structure (embedded resonators, elastic links, metastable elements, etc.) leading to a negative or reverse effective material property. Such a property usually occurs due to a nontrivial collective behavior of many individual structural elements functioning in a synergistic manner. While basic properties of those individual elements can be simple (linear), the interesting collective behavior is owed to a cooperation between the elements achieved with a proper design of their internal structure and interactions. Generally, photonic/electromagnetic, phononic/acoustic and mechanical metamaterials are distinguished [1–34].

Photonic metamaterials are the earliest known class of metamaterials, where the negative effective electrical permittivity (ε) and electromagnetic permeability (μ) are attained simultaneously at the expense of a microscopic superlattice resonator structure leading to a dispersive, nonlinear photonic spectrum. A negative refractive index and the absence of light reflection at an interface of a metamaterial with $\varepsilon < 0$ and $\mu < 0$ and a usual media was first predicted in a 1968 publication [1]. These basic phenomena, in turn, imply amazing practical opportunities, such as superlense and "invisibility cloak" applications [1–7]. One more recent class of photonic metamaterials are plasmonic systems [8–14] featuring nonlinear spectra of collective electron excitation frequencies. Such spectra can provide an efficient generation of higher frequency harmonics, and geometrical and spectral localization of the incident photon energy, being key phenomena for the enhanced photovoltaics, photocatalytic water splitting and other applications [8–14].

Phononic and acoustic metamaterials are dynamical material systems with simple constitutive properties of the individual elements, but counterintuitive collective properties, such as the negative effective mass density and others [15–27]. These unusual collective properties can be realized from the nonlinear vibration frequency spectra, or dispersion relationships $\omega(\mathbf{k})$, where $\mathbf{k}$ is the Fourier wave vector, obtained by applying Fourier transform in time and space to a structural or multi-body dynamics equation. Opportunities for very interesting phenomena, such as acoustic shielding and cloaking arise from the gaps between distinct dispersion branches $\omega_i(\mathbf{k})$, leading to a wide range of important applications ranging from sound insulation and vibration energy harvesting to nondestructive testing and earthquake engineering.

Structural & mechanical metamaterials [28–34,46–48] show *quasistatic* responses to loads that can be interpreted as a negative effective elastic modulus or a negative Poisson ratio by a combination of simple microstructural elements, bars and springs. One recent development [31–34], including by this author and a co-worker [33,34], also uses a structural bistability at the unit cell level that could deploy a polymorphic type phase transformation in the entire material. If properly designed, such a phase transformation can lead to a contraction of the material in the direction of an increasing external load, the negative extensibility

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http://dx.doi.org/10.1016/j.actamat.2016.10.046
1359-6454/

phenomenon. Origami structures provide another interesting type of mechanical metamaterials [46–48] where negative Poisson's ratio, bending and twisting stiffness are either analytically expressed or numerically observed. The work by Silverberg and co-workers [48] is also an example of using bistability of a unit cell to fold reprogrammable mechanical metamaterials based on origami structures.

In this paper, we provide a mathematical foundation and demonstrate the paths toward design and fabrication of a class of structural metamaterials featuring a Reverse Saint-Venant (RSV) edge effect, or shorter the *RSV metamaterials.* When a usual material or structure is loaded statically at surfaces, fine fluctuations of surface strain diminish fast in the material or structure interior with distance to the surface, a phenomenon widely known as the Saint-Venant edge effect, e.g. Refs. [35,36]. In this paper, we have explored some discrete lattices with a higher degree of nonlocality as possible engineered base structures of the RSV metamaterials. In these lattices, certain *coarse* patterns of surface strain may decay faster than the finer ones. More remarkably, such materials will be shown to have an ability to completely block or qualitatively modify certain types of static deformation at surfaces.

The Saint-Venant edge effect is perceived so naturally that one could hardly imagine any violations of it in a material system. Nonetheless, in Section 2, we provide a rigorous proof of concept, followed by discussions of interesting practical opportunities in Section 3, including surface arrest of static deformation and qualitative modification of strain and stress patterns. Conclusions are given in Sections 4. Practical opportunities of the RSV metamaterials are discussed in the context of the density of states calculation, spectral analysis of deformation and phase diagram mapping.

2. Fourier modes of free static deformation

We generally suggest that the Saint-Venant edge effect reversal may occur in materials with internal micro- or mesostructure featuring *nonlocal* interactions. Analysis of these interactions and the resultant effective material properties requires a discrete nonlocal elastic formulation.

Below in this section we suggest a formulation leading directly to the desired metamaterial behavior in a simple but rigorous manner. First we will show that in a local isotropic elastic continuum, only the usual Saint-Venant behavior is possible, and the nonlocality has to be a necessary condition for the sought behavior.

For this discussion, it will be convenient to introduce a nonstandard property of an elastic medium or lattice that we may call the Fourier spectrum of deformation decay parameters, or shorter, *deformation decay spectrum*. This spectrum is a key element of the analytical method offered here, and it exists for any local or nonlocal continuum or a discontinuous structure. In the further discussion (Sections 2.2–2.5) we will see that the Saint-Venant edge effect reversal and surface load arrest phenomena would require occurrence of *asymptotic bandgaps* in the spectrum. In turn, such bandgaps will imply nonlinear and non-monotonous spectral behaviors, only possible in nonlocal media. This spectrum is somewhat analogous, but not similar to the acoustic wave frequency spectrum of a dynamical structure [15–27], where the bandgaps can lead to an acoustic metamaterial.

2.1. Deformation decay spectrum of a continuum solid

Consider a state of plain strain in a *continuum half-plane* for boundary conditions $u(0,y)$ and $v(0,y)$, governed by the homogenous Navier's equations [37] over $x > 0$,

$$
\begin{aligned}
&2(1-\nu)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+(1-2\nu)\frac{\partial^2 u}{\partial y^2}=0,\\
&(1-2\nu)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+2(1-\nu)\frac{\partial^2 v}{\partial y^2}=0
\end{aligned}
\tag{1}
$$

Since any deformation must disappear when $x\to\infty$, it is interesting to consider a fundamental solution as a *decaying Fourier mode* of free static deformation (at real positive η and complex C):

$$
u(x,y)=C_1e^{-\eta x}e^{iqy},\quad v(x,y)=C_2e^{-\eta x}e^{iqy},\quad q\in(-\pi,\pi) \tag{2}
$$

Here, q is a real-valued Fourier parameter of the mode and i is the imaginary unit. When necessary, two independent real-valued solutions can be constructed by taking separately the real and imaginary parts of (2).

Substituting (2) into the governing equation (1) gives two characteristic equations,

$$
\begin{aligned}
&2\eta^2(\nu-1)C_1+iq\eta C_2-q^2(2\nu-1)C_1=0,\\
&\eta^2(1-2\nu)C_2-iq\eta C_1+2q^2(\nu-1)C_2=0
\end{aligned}
\tag{3}
$$

These equations indicate that equation (2) can be a true solution, indeed, but only if $\eta=\pm q$ and $C_2=\mp iC_1$. A solution satisfying specific boundary conditions can be obtained as a superposition of the modes equation (2) with the amplitudes determined by standard Fourier methods. The value η should remain positive for any q, and therefore, we can write

$$
\eta(q)=|q| \tag{4}
$$

This relationship between the decay parameter and the Fourier parameter, Fig. 1, represents the simplest *deformation decay spectrum* of an elastic medium or structure. It applies to any homogenous and isotropic material governed by equation (1). We will also call such a relationship the "η-distribution" below. It will become more sophisticated in the analysis of discrete lattices.

Physical meaning of the decaying mode solution (2,4) can be illustrated as follows. Assume there is a surface traction at $x=0$ leading to the boundary displacements,

$$
u(0,y)=a\cos(qy),\quad v(0,y)=a\sin(qy) \tag{5}
$$

Then, the real amplitude a of these displacements will diminish at $x>0$ with the factor $e^{-|q|x}$, and all strain and stress components will decrease with the same factor as well. Thus, the value $\eta=|q|$ can be interpreted as a basic exponential *decay parameter* of the

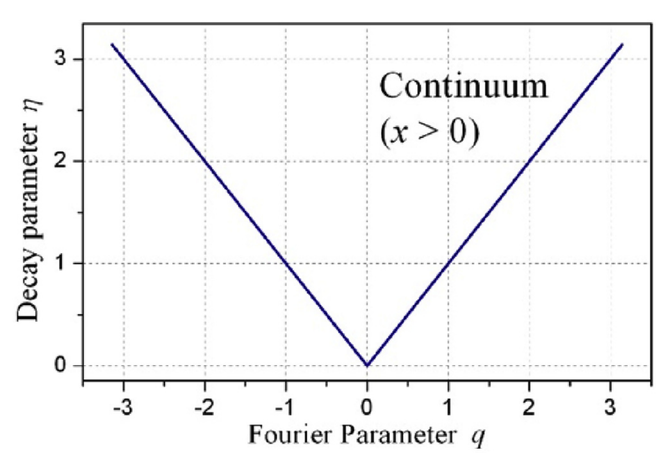

Fig. 1. Deformation decay spectrum (the η-distribution) of the elastic continuum (1).