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Confidence interval estimation of Weibull lower percentiles in small samples via Bayesian inference

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ABSTRACT

Weibull distribution has been vastly used for modeling fracture strength of ceramic and composite materials. Confidence interval estimation of Weibull parameters and percentiles in small samples has been an important concern due to high experimental costs. It was shown previously that in classical inference the Maximum Likelihood Estimation Method is the best method among several alternatives for estimating 95% one-sided confidence lower bounds on the 1st and 10th Weibull percentiles, namely A-basis and B-basis material properties. This study proposes the Bayesian Weibull Method as an alternative using the information that ceramic and composite materials have increasing failure rates, which requires the Weibull shape parameter to be at least 1. Through Monte Carlo simulations, it is shown that the performance of the Bayesian Weibull Method is superior in that it achieves the precision levels of the Maximum Likelihood Estimation Method with significantly smaller sample sizes.

1. Introduction

The Weibull distribution is one of the most popular models for describing variability in the life of manufactured products. In materials science it is widely used for modeling the fracture strength of ceramics, metallic matrix composites and ceramic matrix composites [1], and the flexural strength of brittle materials [2]. It is also used to describe the fracture toughness behavior of steels in ductile-brittle transition region [3]. Let T denote a Weibull random variable modeling the strength of a material. Then T has the following probability density function with parameters σ_0 and m:

$$f(t) = \frac{m}{\sigma_0}\left(\frac{t}{\sigma_0}\right)^{m-1} e^{-\left(\frac{t}{\sigma_0}\right)^m} \tag{1}$$

where $\sigma_0 > 0$ and $m > 0$ are the scale and shape parameters, respectively.

The cumulative distribution function $F(t)$ giving the probability of fracture at a stress level t is expressed as

$$Pr(T \le t) = F(t) = 1 - R(t) = 1 - e^{-\left(\frac{t}{\sigma_0}\right)^m} \tag{2}$$

In reliability studies, a general area of interest is to obtain good estimates of the confidence intervals of Weibull percentiles. In materials science, confidence lower bounds for certain lower percentiles are of particular interest. There are several studies on this particular topic using classical inference methods [4]; however, to our best knowledge, there has been no study investigating the performance of Bayesian inference in the literature. In small samples, it is a known fact that Bayesian methods have substantial advantages over classical methods.

It was shown previously that in classical inference the Maximum Likelihood Estimation (MLE) Method is the best method for estimating 95% one-sided confidence lower bounds on the 1st and 10th Weibull percentiles, namely A-basis and B-basis material properties in materials science [5]. For small samples, this study proposes the Bayesian Weibull (BW) Method as an alternative for which non-informative prior distributions are used for the scale parameter. For the shape parameter, a uniform prior distribution is used with a lower limit of 1 and upper limit of 100. There is strong evidence in the literature (as discussed later) justifying the validity of assuming the shape parameter is within this interval for any ceramic and composite material.

For comparing the performances of the MLE and BW methods, an extensive simulation has been conducted using algorithms developed in C++. In these algorithms, numerical double integration techniques have been employed for BW calculations, and Monte-Carlo simulations are used for the comparisons.

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1.1. A review of the literature

There is a wide variety of methods for estimation of Weibull parameters, percentiles and their confidence intervals. The handbooks [6,7] provide a recent detailed discussion of classical and Bayesian methods. MLE [8], linear regression [9], and weighted linear regression [4,10,11] are among popular classical estimation methods. While there are major differences among classical methods, Bayesian methods basically use the same formulation, as will be discussed later; they differ only by the choice of the prior distributions of the parameters [6,7].

In practice, it is only possible to perform a limited number of experiments. This leads to a small sample of measured strength data, hence low accuracy of estimates. Therefore, statistical properties of the estimators and confidence intervals for the scale and shape parameters have been studied extensively in the materials science literature using classical methods [1–3,9–23].

On the other hand, there are much fewer studies on confidence intervals of Weibull percentiles. In an early study, Bain [24] derived exact confidence intervals of Weibull percentiles using MLE. Later, using the results of this study, Fernandez-Saez et al. [25] conducted a simulation for estimating confidence intervals for the $(100p)$th percentile, t_p. They presented the values of a statistic to be used for computing confidence lower bounds for failure probabilities of $p = 0.01$ and $p = 0.05$ in a tabular form. In a similar study, Barbero et al. [26] showed that linear regression and weighted linear regression methods can also be used for derivation of exact confidence intervals.

Barbero et al. [1] applied the results of [25,26] to A and B-basis material properties. By fitting a curve to the simulation results, they provided an approximate formula for computing the confidence lower bound values for failure probabilities of $p = 0.01$ and 0.10. Birgoren and Dirikolu [27] presented a more efficient and fast simulation tool which allowed the user to determine confidence level, error probability and the number of simulation runs allowing control and minimization of simulation error. Also, Birgoren [4,5] stated that the best method to constuct confidence lower bounds was the one that minimizes false coverage probabilities. With a simulation study, he showed that the MLE is the best method among alternatives including linear and weighted linear regression methods [4]. Phan et al. [28] developed a computer program for computing exact confidence intervals for Weibull parameters and percentiles using Mennon's method. Recently, Edwards et al. [29] presented an induced censoring technique for estimating critical lower percentiles of a failure distribution when the hazard rate reflects multiple aging periods (e.g. a bathtub curve). The practical benefits of the induced censoring technique have been demonstrated by simulation results and practical industrial insights.

Detailed formulations for computing Bayesian confidence lower bounds (or *credible* lower bounds in Bayesian terminology) for Weibull percentiles are provided in [6,7,30]. However, there has been no study comparing their performance with those of classical estimation methods. Furthermore, there has been no study using Bayesian methods for estimating Weibull parameters, percentiles and their confidence intervals in the materials science literature.

1.2. Organization of the study

In this section, we have briefly described the two-parameter Weibull distribution, some of its properties, the purpose of this study and previous studies in the literature. The rest of the paper is arranged as follows. Section 2 describes 95% one-sided confidence lower bounds for Weibull lower percentiles and discusses the classical and Bayesian estimation approaches. The MLE method is explained in Section 3. The BW method is introduced and discussed in Section 4. A simulation study is conducted in Section 5. Finally, a brief discussion of the findings is provided in Section 6.

2. Estimation of confidence lower bounds for weibull percentiles

Estimating lower percentiles in reliability studies is an important issue for manufacturers for evaluation of products' early failures, specification limit improvements, warranty and cost analyses [31]. The $(100p)$th percentile, t_p, corresponding to a predefined failure proabability p, is defined as

$$F\left(t_p\right) = P\left[t \le t_p\right] = p \tag{3}$$

The percentile t_p can be estimated using classical and Bayesian approaches. For both approaches, the estimates of t_p, $\hat{t}_p$, can be quite unreliable, especially in small samples. Therefore, instead of $\hat{t}_p$ values, confidence lower bounds for t_p have been used for the characterization of mechanical properties [4]. In particular, these values corresponding to the 1st and 10th percentiles, estimated with a confidence level of 95%, are known [32] as the *A-basis* and *B-basis material properties*, respectively.

There are two fundamentally different approaches to estimating parameters for statistical approaches: Bayesian and classical inference. Bayesian methods are fundamentally different from the classical ones. The fundamental difference is characterised in the interpretation of probability, definition of the unknown parameters and the usage of prior information. In classical inference it is assumed that the unknown parameters are constant. In contrast, in Bayesian inference a parameter is considered as a random variable whose uncertainty is described by a prior probability distribution [33]. Unlike classical methods, Bayesian methods use the prior knowledge on estimation of a parameter. It combines the prior knowledge and the likelihood function with current observed data using Bayes' theorem to derive a posterior distribution for the model parameter. Confidence intervals of parameters and functions of parameters are calculated from posterior distributions [34].

In the literature, a variety of methods based on classical inference have been developed to estimate confidence lower bounds for Weibull percentiles. The most common methods are MLE, moments, least squares methods and the different modifications of these methods. Since Birgoren [5] showed that the MLE method outperforms the other classical methods in the estimation of A-basis and B-basis material properties, in this study only this method is selected for comparion among classical methods.

3. Maximum likelihood method

The maximum likelihood estimates $\hat{m}$ ve $\hat{\sigma}_0$ for the Weibull parameters m ve σ_0 can be obtained by solving the set of Eqs. (4) and (5) [27].

$$\frac{n}{\hat{m}} - nIn\hat{\sigma}_0 + \sum_{i=1}^{n} t_i - \sum_{i=1}^{n}\left(\frac{t_i}{\hat{\sigma}_0}\right)^{\hat{m}} In\left(\frac{t_i}{\hat{\sigma}_0}\right) = 0 \tag{4}$$

$$\hat{\sigma}_0 = \left(\frac{\sum_{i=1}^{n}(t_i)^{\hat{m}}}{n}\right)^{1/\hat{m}} \tag{5}$$

where $t_1, t_2, \ldots, t_n$ are an observed sample of size n. The Newton-Raphson method is usually employed fo solving Eq. (4) for $\hat{m}$. Then, $\hat{\sigma}_0$ is found by substituting $\hat{m}$ into Eq. (5). Using the values of $\hat{m}$ and $\hat{\sigma}_0$, $\hat{t}_p$ corresponding to a predefined probability of failure p can be obtained by Eq. (6).

$$\hat{t}_p = \hat{\sigma}_0\left[In\left(1/\left(1-p\right)\right)\right]^{1/\hat{m}} \tag{6}$$