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Nonexistence of exact solutions agreeing with the Gaussian beam on the beam axis or in the focal plane

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ABSTRACT

Solutions of the Helmholtz equation which describe electromagnetic beams (and also acoustic or particle beams) are discussed. We show that *an exact solution which reproduces the Gaussian beam waveform on the beam axis does not exist*. This is surprising, since the Gaussian beam is a solution of the paraxial equation, and thus supposedly accurate on and near the beam axis. Likewise, a solution of the Helmholtz equation which exactly reproduces the Gaussian beam in the focal plane does not exist. We show that the last statement also holds for Bessel–Gauss beams. However, solutions of the Helmholtz equation (one of which is discussed in detail) *can* approximate the Gaussian waveform within the central focal region.

1. Introduction

Acoustic, electromagnetic and particle beams are described by solutions of the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi = 0. \tag{1}$$

Eq. (1) results respectively from the linearized hydrodynamic equations, the Maxwell equations, and the Schrödinger equation. The wavenumber $k = \omega/c$ (with ω the angular frequency and c the speed of sound or the speed of light), or is related to the energy per particle, which for the particle mass M is $\hbar^2 k^2/2M$.

The widely used but approximate solution known as the Gaussian beam ([1], Section 16.7, [2], Section 20.3) is

$$\psi_G(x,y,z) = \frac{b}{b+iz}\exp\left\{ikz - \frac{k\left(x^2+y^2\right)}{2(b+iz)}\right\} \tag{2}$$

ψ_G is the fundamental mode solution of the *paraxial equation*, obtained by setting $\psi = e^{ikz}G$ in the Helmholtz equation and then neglecting the term $\partial_z^2 G$ in the resulting equation for G (given below). This amounts to assuming that the dominant z-dependence of the beam lies in the e^{ikz} factor (when propagation is in the z direction, as is assumed here). For axially symmetric solutions we omit the azimuthal derivative, so the Helmholtz equation in cylindrical coordinates, with $\rho = \sqrt{x^2+y^2}$ the distance from the beam axis, takes the form

$$\left(\partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2 + k^2\right)\psi = 0. \tag{3}$$

The substitution $\psi = e^{ikz}G$ gives an equation for G, namely $(\partial_\rho^2 + \rho^{-1}\partial_\rho + 2ik\partial_z + \partial_z^2)G = 0$, in paraxial form $(\partial_\rho^2 + \rho^{-1}\partial_\rho + 2ik\partial_z)G \approx 0$. This paraxial equation has as fundamental solution $G = \frac{b}{b+iz}\exp\left\{-\frac{k\rho^2}{2(b+iz)}\right\}$, thus giving us ψ_G of Eq. (2). Higher modes may be obtained by differentiation of ψ_G with respect to x, y or z, since the differential equations are unchanged by translation in any coordinate. ψ_G depends on the wavenumber k and on the length b, which gives the longitudinal extent of the focal region. The transverse extent in the focal plane is given by $w_0 = \sqrt{2b/k}$. Thus the Gaussian fundamental mode is characterized by a single dimensionless parameter kb. It may seem plausible that when $kb \gg 1$ (focal region large longitudinally compared to k^{-1}) the Gaussian beam would become a satisfactory solution of the Helmholtz equation, everywhere. This is not so: when ψ_G is substituted into $k^{-2}\psi_G^{-1}$ times the Helmholtz equation, we obtain ([2], Section 20.3), instead of zero, $\frac{2}{k^2(b+iz)^2} - \frac{2\rho^2}{k(b+iz)^3} + \frac{\rho^4}{4(b+iz)^4}$. It follows that the errors become small in regions where *both* of the following inequalities hold:

$$k^2\left(b^2+z^2\right) \gg 1 \quad \text{and} \quad b^2+z^2 \gg \rho^2 \tag{4}$$

Fig. 1 shows the modulus and phase of ψ_G, for $kb = 2$. Since the exponent of the modulus of ψ_G tends to $-kb\rho^2/2z^2 = -(kb/2)\tan^2\theta$ far from the origin, the half-angle of the cone of divergence of the Gaussian beam, obtained by setting the exponent equal to -1, is $\theta = \arctan\sqrt{2/kb}$. The divergence angle defined in this way is 45° for $kb = 2$, and 30° for $kb = 6$.

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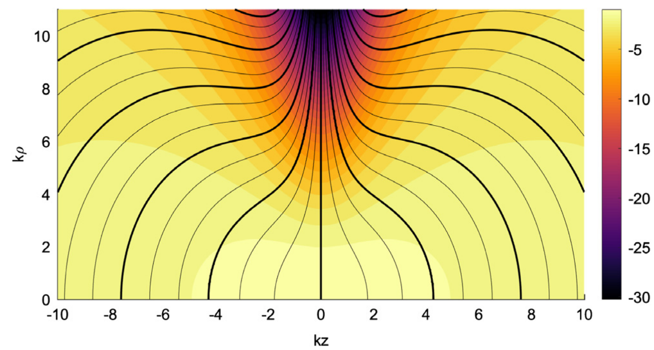

Fig. 1. $\psi_G(\rho,z)$ in the focal region, plotted for $kb=2$, for $k|z|\le 10, k\rho\le 10$. Shading indicates modulus of the wavefunction (logarithmic scale, lighter colour indicates larger modulus). The isophase surfaces are shown at intervals of $\pi/3$. The phase is chosen to be zero at the origin. The isophase contours that are multiples of π are drawn with heavier lines. The three-dimensional picture is obtained by rotating the figure about the beam axis (the horizontal axis). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

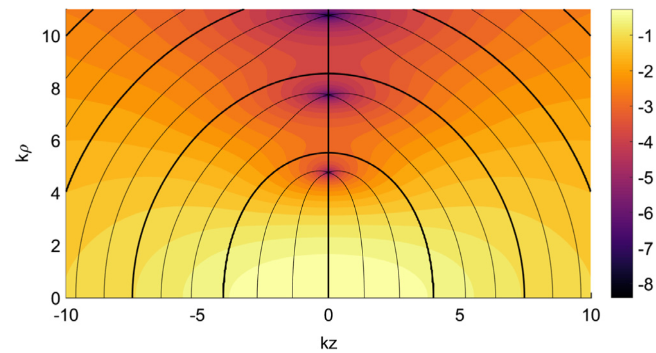

Fig. 2. $\psi_b(\rho,z)$ in the focal region, plotted for $kb=2$, for $k|z|\le 10, k\rho\le 10$. Shading indicates modulus of the wavefunction (logarithmic scale, lighter colour indicates larger modulus). The isophase surfaces are shown at intervals of $\pi/3$. The phase is chosen to be zero at the origin. The isophase contours, other than those that are multiples of π, meet on the zeros of $\psi_0(\rho,z)$, three of which are shown, at $k\rho\approx 4.77, 7.73$ and 10.77. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Many authors [3–10] have investigated methods to build up exact solutions of the Helmholtz equation from solutions of the paraxial equation, typically as expansions in powers of $(kb)^{-1}$ or of $w_0/b=\sqrt{2/kb}$. These expansions have problems, not just in complexity, but in boundedness as well. A case in point is Wünsche's [6] operator method, which aims to get exact solutions from paraxial solutions by acting on the latter with differential operators (given as infinite series of partial derivatives with respect to z). We touch on this method in Appendix C.

Our aim here is different: we ask the question 'can *any* physical solution of the Helmholtz equation duplicate the Gaussian beam on the axis, or in the focal plane?' The answer to both questions is 'no', such solutions do not exist. (By 'physical' is meant causal and having finite beam invariants, as explained in the next Section.) This will be shown in Sections 4 and 5. We also show, in Appendix B, that no Bessel–Gauss beam can be the same in its focal plane as an exact solution. But first we compare and contrast a recent exact solution with the Gaussian beam, in Sections 2 and 3.

2. An exact solution and its properties

A recent paper [11] discusses solutions of the Helmholtz equation (1) which represent transversely bounded beams, of the form

$$\psi(\boldsymbol{r})=e^{im\phi}\int_0^k dq f(k,q)\,J_m\left(\rho\sqrt{k^2-q^2}\right)e^{iqz}. \tag{5}$$

Beams of this form propagate along the z direction. The wave motion is causal [11], meaning that far from the focal region there is no backward propagation. Ref. [11] discusses wavefunctions with no azimuthal dependence ($m=0$), and gives an explicit expression for the case where $f(k,q)$ is proportional to q, in terms of Lommel functions of two variables, or equivalently in terms of products of spherical Bessel and Legendre functions. Proportionality to q at small q is sufficient to ensure the finiteness of beam invariants and of physical quantities such as the energy content per unit length of the beam ([11], Sections 5 and 6).

Wavefunctions of the form (5) can all be generalized to

$$\psi(\boldsymbol{r})=e^{im\phi}\int_0^k dq f(k,q)\,J_m\left(\rho\sqrt{k^2-q^2}\right)e^{iq(z-ib)}. \tag{6}$$

The imaginary translation in z, which leads to the extra factor e^{qb} in the integrand, leaves the Laplacian unchanged, so (6) is still an exact solution of (1).

The $m=0$ beam with $f(k,q)$ set equal to a constant in (6) does approximate ψ_G uniformly along the beam axis, with error of order e^{-kb}. However, $f(k,q)=$ constant is not physically possible: it does not give a finite energy content per unit length of the beam. For example, the corresponding transverse magnetic (TM) beam has energy content per unit length of the beam ([12], Eq. (26)) proportional to $\int_0^k dq q^{-1}\left(k^2-q^2\right)e^{2qb}$, which diverges logarithmically.

As an example of a beam waveform which has all of the required physical properties, we shall consider the wavefunction of Section 9 of [11]:

$$\psi_b(\rho,z)=\frac{b^2}{\left[e^{kb}(kb-1)+1\right]}\int_0^k dq\, q\, e^{q(b+iz)}J_0\left(\rho\sqrt{k^2-q^2}\right). \tag{7}$$

The prefactor in (7) normalizes the wavefunction to unity at the origin $\rho=0, z=0$, for easier comparison with ψ_G given in (2), which is also normalized to unity at the origin.

We shall show that, for $e^{kb}\gg 1$ and $\rho^2\ll b/k$, where the Gaussian waveform has some validity, the wavefunction ψ_b corresponds closely to it, provided that also $|z|\ll kb^2$. There are no constraints on where ψ_b may be used, being an exact solution of (1). Fig. 2 shows $\psi_b(\rho,z)$ in the focal region around the origin, for kb again set equal to 2.

On the beam axis $\rho=0$ we have

$$\begin{aligned}\psi_b(0,z)&=\frac{b^2}{\left[e^{kb}(kb-1)+1\right]}\int_0^k dq\, q\, e^{q(b+iz)}\\&=\left(\frac{b}{b+iz}\right)^2\frac{e^{k(b+iz)}\left[k(b+iz)-1\right]+1}{\left[e^{kb}(kb-1)+1\right]}.\end{aligned} \tag{8}$$

An explicit form of ψ_b at a general point (ρ,z) was found in [11], using the fact that the expression (7) is a cylindrically symmetric non-singular solution of the Helmholtz equation, and may thus be expanded as a sum over products of Legendre polynomials and spherical Bessels,

$$\begin{aligned}\psi_b(\rho,z)&=\frac{(kb)^2}{\left[e^{kb}(kb-1)+1\right]}\sum a_n P_n\left(\frac{z-ib}{R}\right)j_n(kR),\\R&=(z-ib)\sqrt{1+\rho^2/(z-ib)^2}.\end{aligned} \tag{9}$$

As in Ref. [11], R is chosen as a branch of the complex radial coordinate resulting from an imaginary displacement along the beam axis:

$$r=\sqrt{\rho^2+z^2}\to R=\sqrt{\rho^2+(z-ib)^2}. \tag{10}$$

The coefficients a_n in the expansion are given in [11], Appendix B. There is only one non-zero odd coefficient, $a_1=2i$. The even coefficients we shall rename as $a_{2n}=A_n$, so that

$$\begin{aligned}\psi_b(\rho,z)=&\frac{(kb)^2}{\left[e^{kb}(kb-1)+1\right]}\left\{2iP_1\left(\frac{z-ib}{R}\right)j_1(kR)\right.\\&\left.+\sum_0^\infty A_nP_{2n}\left(\frac{z-ib}{R}\right)j_{2n}(kR)\right\}.\end{aligned} \tag{11}$$