Optics Communications 405 (2017) 132–142

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Finite Gaussian wavelet superposition and Fresnel diffraction integral for calculating the propagation of truncated, non-diffracting and accelerating beams

Moisés Cywiak [a,*], David Cywiak [b], Etna Yáñez [a]

[a] Centro de Investigaciones en Óptica A. C. Loma del Bosque 115, Colonia Lomas del Campestre León, Guanajuato, México, C. P. 371506, Mexico
[b] Centro Nacional de Metrología. km 4.5 Carretera a Los Cués , Municipio El Marqués, Querétaro., México, C.P. 76246, Mexico

ARTICLE INFO

Keywords:
Fresnel diffraction integral
Diffraction theory
Accelerating beams
Superposition of Gaussian wavelets

ABSTRACT

We demonstrate that the propagation of truncated, non-diffracting and accelerating beams can be accurately calculated by using a method that represents these beams by a finite superposition of Gaussian wavelets to be in turn propagated by means of the Fresnel diffraction integral. We support our proposal by demonstrating analytically that the Fresnel diffraction integral describes properly the non-diffracting and accelerating characteristics of non-truncated beams under propagation. We present numerical results of the propagation of this type of truncated beams and we propose analytical equations of a truncated accelerating beam with improved performance.

0. Introduction

The non-diffracting zero-order Bessel beam of the first kind was analytically obtained in [1] as a solution to the Helmholtz differential equation in cylindrical coordinates with radial symmetry. Its properties under propagation have been studied in [2–4]. Non-diffracting properties were then extended to the nth-order Bessel beam of the first kind in [5] and compared with other non-diffracting beams. Propagation equations obtained as solutions of the Helmholtz differential equation were obtained for the special case of non-diffracting beams in [6,7]. Additionally, by considering the Helmholtz equation in a paraxial approximation, a Schrödinger-type solution consisting of an Airy beam which is referred to as an accelerating beam since it travels in bending trajectories and being also non-diffracting has been studied in [8,9]. Fresnel diffraction propagation of Airy beams has been investigated in [10] by calculating their propagation using the Cornu spiral.

A non-diffracting beam physically realizable is spatially limited. It can be represented by an ideal beam, which is truncated by means of an appropriate aperture. Its non-diffracting range decreases and analytical equations in an exact (closed) form to calculate its propagation cannot be obtained in general, hence, requiring numerical methods. An efficient numerical method to calculate the propagation of complex fields using a finite superposition of Gaussian wavelets (FSGW) has been reported in [11,12]. However, before using this method, it must be determined whether its applicability can be extended for calculating the propagation of non-diffracting and accelerating beams. For this, the following aspects must be taking into consideration. First, Gaussian beams widen upon propagation making it questionable whether such a superposition is appropriate for propagating non-diffracting beams. Second, as the Fresnel diffraction integral is not an exact analytical solution to the Helmholtz differential equation but only a paraxial approximation, it must be found whether this integral preserves non-diffracting beams under propagation and, furthermore, whether this integral is useful for accurately calculating the bending trajectory of accelerating beams.

The above apparent limitations are overcome as follows. First, relying on the linearity of the Fresnel diffraction integral, if the beam to be propagated is properly represented by a FSGW, then, the linearity of this integral implies that the propagated field should also be accurate regardless of the widening of the Gaussian wavelets in the superposition. The second apparent limitation is overcome by analytically demonstrating that the Fresnel diffraction integral, in fact, preserves accurately the non-diffracting and accelerating properties of the beams analyzed in this report.

In the following sections, we demonstrate that the Fresnel diffraction integral preserves non-diffracting, nth-order Bessel beams of the first kind as well as accelerating Airy beams when both of these extend spatially without any truncation. Under this condition, exact analytical

* Corresponding author.
E-mail address: moi@cio.mx (M. Cywiak).

http://dx.doi.org/10.1016/j.optcom.2017.08.015
Received 19 April 2017; Received in revised form 24 July 2017; Accepted 8 August 2017
Available online 23 August 2017

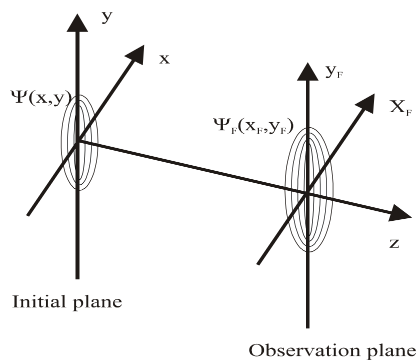

Fig. 1. Propagation of a field from an initial plane with coordinates (x, y) up to an observation plane with coordinates (x_F, y_F). The planes are parallel to each other and separated a distance z. The amplitude distributions are $\Psi(x, y)$ and $\Psi_F(x_F, y_F)$ respectively.

equations of propagation can be obtained. Based upon these results, we now extend the applicability of our FGSW method to propagate truncated beams for which analytical equations cannot be expressed in closed form. We present numerical examples performed by means of the FSGW method. Finally, motivated on the Airy beam, we propose a new truncated beam that in addition of showing comparable accelerating properties exhibits improved performance.

1. Analytical description

Fig. 1 illustrates the physical situation.

At an initial plane with coordinates (x, y), a field with amplitude distribution $\Psi(x, y)$ propagates towards an observation plane with coordinates (x_F, y_F) located at a distance z and being parallel to the initial plane. At the initial plane the spatial coordinates (x, y) will be represented in cylindrical form as,

$$x = \rho\cos(\varphi);\ \ y = \rho\sin(\varphi), \tag{1}$$

analogously at the observation plane,

$$x_F = \rho_F\cos(\varphi_F);\ \ y_F = \rho_F\sin(\varphi_F). \tag{2}$$

At the observation plane the amplitude distribution $\Psi_F(x_F, y_F)$ is calculated by means of the Fresnel diffraction integral [13],

$$\Psi_F(x_F, y_F) = \frac{\exp\left(i\frac{2\pi}{\lambda}z\right)}{i\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(x, y)\exp\left\{i\frac{\pi}{\lambda z}\left[\left(x - x_F\right)^2 + \left(y - y_F\right)^2\right]\right\}dx\,dy. \tag{3}$$

In Eq. (3) λ is the wavelength of the field.

In the following subsection Eqs. (1)–(3) will be applied to calculate the propagation of non-truncated beams.

1.1. *Propagation of non-truncated, non-diffracting, nth-order Bessel beams of the first kind*

In this subsection, we analytically demonstrate that the Fresnel diffraction integral preserves non-truncated, non-diffracting nth-order Bessel beams of the first kind by writing the amplitude distribution at the initial plane as $\Psi(\rho, \varphi) = J_n(a\rho)\exp(in\varphi)$. For this case the Fresnel diffraction integral (3) expressed in cylindrical coordinates reads,

$$\Psi_F\left(\rho_F\right) = 2\pi(i)^n\frac{\exp\left(i\frac{2\pi}{\lambda}z\right)}{i\lambda z}\exp\left(i\frac{\pi}{\lambda z}\rho_F^2\right)\exp\left(in\,\varphi_F\right) \times\int_0^{\infty}\exp\left(i\frac{\pi}{\lambda z}\rho^2\right)J_n(a\rho)\,J_n\left(-\frac{2\pi}{\lambda z}\rho_F\rho\right)\rho\,d\rho. \tag{4}$$

The integral in Eq. (4) can be calculated using Weber's integral [14–16] expressed as,

$$\int_0^{\infty}\exp\left(-s^2t^2\right)J_\nu(at)J_\nu(bt)t\,dt = \frac{1}{2s^2}\exp\left(-\frac{a^2+b^2}{4s^2}\right)i^{-\nu}J_\nu\left(i\frac{ab}{2s^2}\right). \tag{5}$$

Eq. (5) allows to write Eq. (4) as,

$$\Psi_F\left(\rho_F\right) = \exp\left(i\frac{2\pi}{\lambda}z\right)\exp\left(-i\frac{\lambda a^2}{4\pi}z\right)J_n\left(a\rho_F\right)\exp\left(in\,\varphi_F\right). \tag{6}$$

Eq. (6) demonstrates that the Fresnel diffraction integral preserves non-truncated, non-diffracting nth-order Bessel beams of the first kind and additionally provides their appropriate phase terms as a function of distance of propagation.

1.2. *Schrödinger-type beams*

To establish the Schrödinger-type differential equation as a paraxial approximation of the Helmholtz differential, we consider the two-dimensional Helmholtz equation as,

$$\frac{\partial^2\Psi(x,z)}{\partial x^2} + \frac{\partial^2\Psi(x,z)}{\partial z^2} + k_0^2\Psi(x,z) = 0. \tag{7}$$

In Eq. (7) $k_0 = 2\pi/\lambda$, as usual.

Now a solution of the form, $\Psi(x,z) = \Psi_A(x,z)\exp\left(ik_0z\right)$ is proposed and substituted in Eq. (7), giving,

$$\frac{\partial^2\Psi_A(x,z)}{\partial x^2} + \frac{\partial^2\Psi_A(x,z)}{\partial z^2} + i2k_0\frac{\partial\Psi_A(x,z)}{\partial z} = 0. \tag{8}$$

For the paraxial approximation, the second partial derivative with respect to z is neglected in Eq. (8). Thus,

$$\frac{\partial^2\Psi_A(x,z)}{\partial x^2} + i2k_0\frac{\partial\Psi_A(x,z)}{\partial z} = 0. \tag{9}$$

Eq. (9) is the well-known analytical expression for the paraxial approximation of the two-dimensional Helmholtz differential equation.

We will now demonstrate that, aside from the phase term $\exp\left(ik_0z\right)$, the Fresnel diffraction integral is an exact solution of the Schrödinger-type differential equation (9). For this, we write the Fourier transform of $\Psi_A(x,z)$ and its corresponding inverse transform $\Omega_A(u,z)$ as,

$$\Omega_A(u,z) = \int_{-\infty}^{\infty}\Psi_A(x,z)\exp(-i2\pi ux)\,dx;$$
$$\Psi_A(x,z) = \int_{-\infty}^{\infty}\Omega_A(u,z)\exp(i2\pi ux)\,du. \tag{10}$$

In Eq. (10) u corresponds to the spatial frequency of x.

Eq. (10) allows to rewrite (9) as,

$$-4\pi^2u^2\Omega_A(u,z) + i2k_0\frac{\partial\Omega_A(u,z)}{\partial z} = 0. \tag{11}$$

Eq. (11) is solved straightforward as,

$$\Omega_A(u,z) = A(u)\exp\left(-i\pi\lambda zu^2\right). \tag{12}$$

In Eq. (12) when $z = 0$ gives, $A(u) = \Omega_A(u,0)$. Thus, (12) can be written as,

$$\Omega_A(u,z) = \Omega_A(u,0)\exp\left(-i\pi\lambda zu^2\right). \tag{13}$$

From Eq. (13) the transfer function $T_A(u)$ of the free space propagation can be recognized as,

$$T_A(u) = \exp\left(-i\pi\lambda zu^2\right). \tag{14}$$

Calculating the inverse Fourier transform of (14) gives the impulse response as,

$$I_A(x,z) = \frac{1}{\sqrt{i\lambda z}}\exp\left(i\frac{\pi}{\lambda z}x^2\right). \tag{15}$$