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Invited paper

Probability density functions of instantaneous Stokes parameters on weak scattering

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A R T I C L E I N F O

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A B S T R A C T

The single-point probability density functions (PDF) of the instantaneous Stokes parameters of a polarized plane-wave light field scattered from a three-dimensional, statistically stationary, weak medium with Gaussian statistics and Gaussian correlation function have been studied for the first time. Apart from the scattering geometry the PDF distributions of the scattered light have been related to the illumination's polarization state and the correlation properties of the medium.

1. Introduction

An electromagnetic, wide-sense statistically stationary beam-like light field may be characterized at a single position in space by a set of four instantaneous Stokes parameters [1] being the generalizations of the classic average Stokes parameters [2]. In general, the average Stokes parameters, being the second-order moments of the electromagnetic field are not related to its higher-order moments and its Probability Density Functions [PDF]. However, in the case when the field at a single point obeys Gaussian statistics the higher-order moments and the PDFs of the instantaneous Stokes parameters can be expressed via the average Stokes parameters by a set of analytical formulas [3–5].

Free-space propagation of the instantaneous Stokes parameters of a beam generated by a planar source of Gaussian Schell-model type with Gaussian single-point statistics has been examined in Ref. [6]. The PDFs of the instantaneous Stokes parameters have been shown to preserve their original structure but to acquire sharper profiles with the growing propagation distance from the source. Special interest has been paid to the second-order moment of the fluctuating intensity [7,8], known as the scintillation index. The changes in some statistics of the instantaneous Stokes parameters on propagation in the atmospheric turbulence, in optical fibers and on scattering from rough surfaces have also been examined [9–11].

Scattering of light from deterministic or random particles is yet another situation in which the statistical properties of illumination may be modified [12]. In particular, when a deterministic optical field is scattered from a random medium the generated field itself becomes random, acquiring the statistics of the medium. The average Stokes parameters of an electromagnetic plane wave scattered by a random particle have been investigated in [13] (see also [14] and [15] for an alternative representation of an electromagnetic field via the 2×2 correlation matrix). The intensity–intensity correlations of light on scattering from media with Gaussian statistics have been examined in Refs. [16–22]. However, the moments of arbitrary order and the PDFs of the instantaneous Stokes parameters of scattered light have not been discussed so far.

In this paper we analyze the distributions of the PDFs of the instantaneous Stokes parameters of an electromagnetic, polarized plane wave on scattering from a weak, stationary, homogeneous and isotropic medium of Schell-model type with Gaussian correlation function [23]. More specifically we study how such distributions vary as a function of polar and azimuthal scattering angles, polarization state of illumination and the two parameters of the scatterer: typical width and typical correlation width. We will confine our attention only to far-field scattering, where the electromagnetic field is transverse, i.e., two-dimensional, and can be characterized by four local Stokes parameters, unlike in other regions where it might require complete three-dimensional polarimetric characterization.

2. Scattering of the average and instantaneous Stokes vectors

We begin by outlining the scattering scenario (see Fig. 1). Consider electric vector-field $\mathbf{E}^{(i)}(\mathbf{r}') = [E_x^{(i)}(\mathbf{r}'), E_y^{(i)}(\mathbf{r}')]$ incident on a scatterer occupying domain D at position $\mathbf{r}'$. Here x and y denote two mutually orthogonal polarization directions transverse to the scattering axis z. We also imply that the field is polychromatic but suppress its dependence

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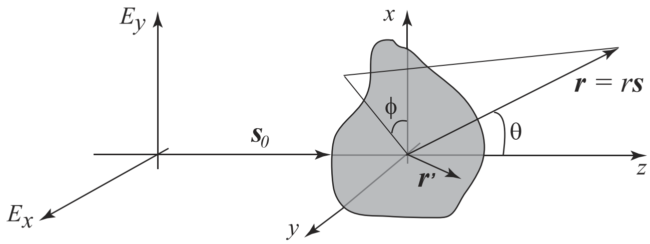

Fig. 1. Illustrating the notation.

on frequency for brevity. The average generalized Stokes vector of the incident light has form [24]

$$\begin{aligned}
\langle S_0^{(i)}(\mathbf{r}_1', \mathbf{r}_2')\rangle &= \langle E_x^{(i)*}(\mathbf{r}_1')E_x^{(i)}(\mathbf{r}_2')\rangle + \langle E_y^{(i)*}(\mathbf{r}_1')E_y^{(i)}(\mathbf{r}_2')\rangle,\\
\langle S_1^{(i)}(\mathbf{r}_1', \mathbf{r}_2')\rangle &= \langle E_x^{(i)*}(\mathbf{r}_1')E_x^{(i)}(\mathbf{r}_2')\rangle - \langle E_y^{(i)*}(\mathbf{r}_1')E_y^{(i)}(\mathbf{r}_2')\rangle,\\
\langle S_2^{(i)}(\mathbf{r}_1', \mathbf{r}_2')\rangle &= \langle E_x^{(i)*}(\mathbf{r}_1')E_y^{(i)}(\mathbf{r}_2')\rangle + \langle E_y^{(i)*}(\mathbf{r}_1')E_x^{(i)}(\mathbf{r}_2')\rangle,\\
\langle S_3^{(i)}(\mathbf{r}_1', \mathbf{r}_2')\rangle &= i[\langle E_y^{(i)*}(\mathbf{r}_1')E_x^{(i)}(\mathbf{r}_2')\rangle - \langle E_x^{(i)*}(\mathbf{r}_1')E_y^{(i)}(\mathbf{r}_2')\rangle],
\end{aligned} \tag{1}$$

where star stands for complex conjugate and angular brackets denote the ensemble average taken over monochromatic field realizations.

Let the scatterer occupy domain D of a three-dimensional space and be characterized by the scattering potential

$$F(\mathbf{r}') = \frac{k^2}{4\pi}[n^2(\mathbf{r}') - 1], \tag{2}$$

$n(\mathbf{r}')$ being the refractive index at position $\mathbf{r}'$ within the scatterer, $k = 2\pi/\lambda$, is the wave-number of light, λ being its wavelength in vacuum. Further, let the potential obey Gaussian statistics [23] and have two-point correlation function

$$C_F(\mathbf{r}_1', \mathbf{r}_2') = \langle F(\mathbf{r}_1')F(\mathbf{r}_2')\rangle_M, \tag{3}$$

where angular brackets with subscript M stands for average taken over the scatterer's realizations.

Within the validity of the first Born approximation the three-dimensional scattered electromagnetic field along direction $\mathbf{r} = r\mathbf{s}$, where $\mathbf{s}$ is a unit vector along the direction of vector $\mathbf{r}$, can be found from double cross product [27]:

$$\mathbf{E}^{(s)}(r\mathbf{s}) = -\mathbf{s} \times \left[\mathbf{s} \times \int_D F(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\mathbf{E}^{(i)}(\mathbf{r}')d^3r'\right], \tag{4}$$

where

$$G(\mathbf{r}, \mathbf{r}') = \frac{\exp[ik|\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|} \tag{5}$$

is the free-space Green's function. More explicitly, three Cartesian components of the scattered field have the form:

$$\begin{aligned}
E_x^{(s)}(r\mathbf{s}) &= (1 - s_x^2)Q_x(r\mathbf{s}) - s_x s_y Q_y(r\mathbf{s}),\\
E_y^{(s)}(r\mathbf{s}) &= -s_x s_y Q_x(r\mathbf{s}) + (1 - s_y^2)Q_y(r\mathbf{s}),\\
E_z^{(s)}(r\mathbf{s}) &= -s_x s_z Q_x(r\mathbf{s}) - s_y s_z Q_y(r\mathbf{s}),
\end{aligned} \tag{6}$$

where

$$Q_i(r\mathbf{s}) = \int_D F(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')E_i^{(i)}(\mathbf{r}')d^3r', \quad (i = x, y). \tag{7}$$

In spherical coordinate system (r, θ, ϕ) associated with the scatterer (see Fig. 1) the coordinates of the unit vector $\mathbf{s}$ may be expressed as

$$s_x = \sin\theta\cos\phi, \quad s_y = \sin\theta\sin\phi, \quad s_z = \cos\theta, \tag{8}$$

where θ and ϕ are the azimuthal and polar angles, respectively. In the far zone of the scatterer the field is transverse with respect to scattering direction $\mathbf{s}$ and hence can be characterized only by the azimuthal and polar components, E_θ and E_ϕ, in the spherical coordinates (see also [13,25]):

$$\begin{aligned}
E_\theta^{(s)}(r\mathbf{s}) &= \cos\theta\cos\phi E_x^{(s)}(r\mathbf{s}) + \cos\theta\sin\phi E_y^{(s)}(r\mathbf{s}) - \sin\theta E_z^{(s)}(r\mathbf{s}),\\
E_\phi^{(s)}(r\mathbf{s}) &= -\sin\phi E_x^{(s)}(r\mathbf{s}) + \cos\phi E_y^{(s)}(r\mathbf{s}).
\end{aligned} \tag{9}$$

On combining Eqs. (6) and (9) we find, after some trigonometric manipulations, that

$$\begin{aligned}
E_\theta^{(s)}(r\mathbf{s}) &= \cos\theta\cos\phi Q_x(r\mathbf{s}) + \cos\theta\sin\phi Q_y(r\mathbf{s}),\\
E_\phi^{(s)}(r\mathbf{s}) &= -\sin\phi Q_x(r\mathbf{s}) + \cos\phi Q_y(r\mathbf{s}),
\end{aligned} \tag{10}$$

where Q_x and Q_y are given in Eq. (7).

The average Stokes parameters of the transverse scattered field in the spherical coordinates can be defined as [13]

$$\langle S_0^{(s)}(r\mathbf{s})\rangle = \langle E_\theta^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s})\rangle + \langle E_\phi^{(s)*}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s})\rangle, \tag{11a}$$

$$\langle S_1^{(s)}(r\mathbf{s})\rangle = \langle E_\theta^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s})\rangle - \langle E_\phi^{(s)*}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s})\rangle, \tag{11b}$$

$$\langle S_2^{(s)}(r\mathbf{s})\rangle = \langle E_\theta^{(s)*}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s})\rangle + \langle E_\phi^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s})\rangle, \tag{11c}$$

$$\langle S_3^{(s)}(r\mathbf{s})\rangle = i[\langle E_\phi^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s})\rangle - \langle E_\theta^{(s)}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s})\rangle]. \tag{11d}$$

On substituting from Eqs. (10) into Eqs. (11) and using the far-zone approximation of the Green's function,

$$G(r\mathbf{s}, \mathbf{r}') \approx \frac{\exp[-ik\mathbf{s}\cdot\mathbf{r}' + ikr]}{r}, \tag{12}$$

we obtain for the average scattered Stokes parameters the formulas:

$$\langle S_0^{(s)}(r\mathbf{s})\rangle = (\cos^2\theta + 1)T_0(r\mathbf{s}) - \sin^2\theta\cos 2\phi T_1(r\mathbf{s}) - \sin^2\theta\sin 2\phi T_2(r\mathbf{s}), \tag{13a}$$

$$\begin{aligned}
\langle S_1^{(s)}(r\mathbf{s})\rangle = &-\sin^2\theta T_0(r\mathbf{s}) + \cos 2\phi(1 + \cos^2\theta)T_1(r\mathbf{s})\\
&+ \sin 2\phi(1 + \cos^2\theta)T_2(r\mathbf{s}),
\end{aligned} \tag{13b}$$

$$\langle S_2^{(s)}(r\mathbf{s})\rangle = -2\cos\theta\sin 2\phi T_1(r\mathbf{s}) + 2\cos\theta\cos 2\phi T_2(r\mathbf{s}), \tag{13c}$$

$$\langle S_3^{(s)}(r\mathbf{s})\rangle = 2\cos\theta T_3(r\mathbf{s}), \tag{13d}$$

where

$$\begin{aligned}
T_\alpha(r\mathbf{s}) = &\frac{1}{2r^2}\int_D\int_D \langle S_\alpha^{(i)}(\mathbf{r}_1', \mathbf{r}_2')\rangle C_F(\mathbf{r}_1', \mathbf{r}_2')\\
&\times\exp[-ik(\mathbf{s}\cdot(\mathbf{r}_2' - \mathbf{r}_1'))]d^3r_1'd^3r_2',\\
&(\alpha = 0, 1, 2, 3).
\end{aligned} \tag{14}$$

Eqs. (13)–(14) imply that while Stokes parameter $\langle S_3(r\mathbf{s})\rangle$ of the scattered field depends solely on the corresponding generalized Stokes parameter $\langle S_3^{(i)}(\mathbf{r}_1', \mathbf{r}_2')\rangle$ of the incident field the other three Stokes parameters are coupled. This situation is different from free-space paraxial propagation where all four parameters propagate independently [24].

The instantaneous Stokes parameters of the scattered far field can be defined as [1]

$$S_0^{(s)}(r\mathbf{s}) = E_\theta^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s}) + E_\phi^{(s)*}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s}), \tag{15a}$$

$$S_1^{(s)}(r\mathbf{s}) = E_\theta^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s}) - E_\phi^{(s)*}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s}), \tag{15b}$$

$$S_2^{(s)}(r\mathbf{s}) = E_\theta^{(s)*}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s}) + E_\phi^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s}), \tag{15c}$$

$$S_3^{(s)}(r\mathbf{s}) = i[E_\phi^{(s)*}(r\mathbf{s})E_\theta^{(s)}(r\mathbf{s}) - E_\theta^{(s)}(r\mathbf{s})E_\phi^{(s)}(r\mathbf{s})]. \tag{15d}$$

The single-point PDFs of the instantaneous Stokes parameters of the stationary light field governed by Gaussian statistics have been derived in Refs. [3–5] and have forms:

$$\begin{aligned}
p[S_0^{(s)}(r\mathbf{s})] = &\frac{1}{P^{(s)}(r\mathbf{s})\langle S_0^{(s)}(r\mathbf{s})\rangle}\left\{\exp\left[-\frac{2S_0^{(s)}(r\mathbf{s})}{(1 + P^{(s)}(r\mathbf{s}))\langle S_0^{(s)}(r\mathbf{s})\rangle}\right]\right.\\
&\left.-\exp\left[-\frac{2S_0(r\mathbf{s})}{(1 - P^{(s)}(r\mathbf{s}))\langle S_0^{(s)}(r\mathbf{s})\rangle}\right]\right\},
\end{aligned} \tag{16}$$

$$\begin{aligned}
p[S_\alpha^{(s)}(r\mathbf{s})] = &\frac{1}{\sqrt{\langle S_0^{(s)}(r\mathbf{s})\rangle^2(1 - P^{(s)}(r\mathbf{s})^2) + \langle S_\alpha^{(s)}(r\mathbf{s})\rangle^2}}\\
&\times\exp\left[2\frac{S_\alpha^{(s)}(r\mathbf{s})\langle S_\alpha^{(s)}(r\mathbf{s})\rangle}{\langle S_0^{(s)}(r\mathbf{s})\rangle^2(1 - P^{(s)}(r\mathbf{s})^2)}\right]\\
&\times\exp\left[-2\frac{|S_\alpha^{(s)}(r\mathbf{s})|\sqrt{\langle S_0(r\mathbf{s})\rangle^2(1 - P^{(s)}(r\mathbf{s})^2) + \langle S_\alpha^{(s)}(r\mathbf{s})\rangle^2}}{\langle S_0^{(s)}(r\mathbf{s})\rangle^2(1 - P^{(s)}(r\mathbf{s})^2)}\right]\\
&(\alpha = 1, 2, 3).
\end{aligned} \tag{17}$$