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# Modulational instability in nonlocal Kerr media with a sine-oscillatory response

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ABSTRACT

We discuss modulational instability (MI) in nonlocal optical Kerr media with a sine-oscillatory response which can model nematic liquid crystals with negative dielectric anisotropy. In the framework of nonlocal nonlinear Schrödinger equation, MI in this type of media is found to have two unique properties different from those in other media discussed previously. First, MI exists both when the Kerr coefficient is positive and when it is negative. Second, the maximum gain points of MI do not shift with light intensity. We also explore the physical mechanism behind MI in local and nonlocal optical Kerr media by utilizing the theory of four-wave mixing. Through introducing a phase mismatch term ($\Delta k$) and a growth factor ($\gamma$), we deduce that the necessary and sufficient condition for MI to occur is that the phase mismatching during the four-wave-mixing process should be small enough such that $|\Delta k| < 2|\gamma|$. Based on this condition, we can uniformly and consistently explain the results of MI in optical Kerr media obtained in the current work as well as those presented in previous work by others.

## 1. Introduction

Modulational instability (MI) is a type of ubiquitous instability found in nonlinear systems. It signifies the amplification of random perturbations in a harmonic wave when it propagates in nonlinear media. The growth of the perturbations generates spectral sidebands, causing the harmonic wave to evolve into a modulated state. MI has been studied and identified in various physical systems, such as fluids [1], plasma [2], and nonlinear optics [3–7], among others. In the context of optical fibers, MI causes a continuous wave to break up into a pulse train with a high repetition rate [5,6] in general. For a monochromatic spatial optical beam, MI splits the beam into a transversely periodic array of fine localized structures (filamentation) [3]. Therefore, in optical Kerr media, self-phase-induced MI is considered to be the precursor of bright soliton formation, whereas dark solitons require the absence of such MI. However, dark-soliton-like pulse-trains can be generated [8,9] by cross-phase-induced MI [10]. There are two equivalent methods to discuss the MI in nonlinear optics. One of these is to examine the evolution of perturbations with linear stability analysis in the framework of nonlinear Schrödinger equation [3,7]. The other is to interpret the MI in terms of the four-wave-mixing (FWM) process [7,11,12].

Spatial nonlocality is one of the most important properties of the optical Kerr effect. The nonlocality means that the light-induced nonlinear refractive index (NRI) at a given point is determined not only by the light intensity at that point but also by the light intensity near that point. This can be described phenomenologically as [13]

$$\Delta n = n_2 \int_{-\infty}^{\infty} R(\mathbf{r}_\perp - \mathbf{r}'_\perp)|E(\mathbf{r}'_\perp, z)|^2 \mathrm{d}\mathbf{r}'_\perp, \tag{1}$$

where $\Delta n$ is the NRI, $n_2$ is the Kerr coefficient determined by material properties, $\mathbf{r}_\perp$ is the transverse coordinate vector, the real symmetric function $R(\mathbf{r}_\perp)$ is the response function of the nonlocal optical Kerr media, and $E$ is the optical field. Since Snyder and Mitchell brought spatial nonlocality in focus [14], research on nonlocal spatial optical solitons have been systematically conducted in optical Kerr media with strong nonlocality, including nematic liquid crystals (NLC) with positive dielectric anisotropy [15–20] and lead glasses [21–25], among others. Recently, bright optical solitons were observed in the planar cell containing NLC with negative dielectric anisotropy [26]. In the 1+1 dimensional model of this system, the NRI satisfies the equation below [26,27]

$$w_m^2 \frac{\mathrm{d}^2 \Delta n}{\mathrm{d}x^2} + \Delta n = n_2 |E|^2, \tag{2}$$

where $w_m$ is a positive constant representing the nonlinear characteristic length, and the Kerr coefficient $n_2$ is negative. The response function derived from equation (2) is sine-oscillatory.

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$$R(x) = \frac{1}{2w_m}\sin\left(\frac{|x|}{w_m}\right). \tag{3}$$

This response function was obtained at first in the model of quadratic solitons [28]. However, the nonlinear process of quadratic solitons is a second-order nonlinear effect rather than one of the third order. Both bright and dark solitons have been found in optical Kerr media with the NRI described by Eq. (2) [27]; however, the dark solitons are unstable. Hence, discussing MI in the spatially nonlocal optical Kerr media with the sine-oscillatory response function is useful to the understanding of the bright soliton stability and the instability of the dark solitons in such a system.

In spatially nonlocal nonlinear systems, the nonlocality has strong influences on MI. The first theoretical discussion of MI in spatially nonlocal optical Kerr media [29,30] was based on linear stability analysis in the framework of nonlocal nonlinear Schrödinger equation (NNLSE), and it showed that the gain spectrum of MI is affected by the characteristic length of the nonlocality and the profile of the response function. Later, Wyller et al. discussed MI in the model of quadratic solitons with the sine-oscillatory response [31] using the same method. It can be shown from Wyller's work that the effective NRI, which is the second harmonic optical field, is induced by the square of the fundamental wave ($E^2$), while in spatially nonlocal optical Kerr media, the NRI is induced by the light intensity ($|E|^2$), as shown in Eq. (1). Therefore, MI in spatially nonlocal optical Kerr media with the sine-oscillatory response function can be different from that in the model of quadratic solitons.

In this paper, we analytically study MI in spatially nonlocal optical Kerr media with a sine-oscillatory response, and we discuss the physical mechanism behind the MI. The contents of the paper are organized as follows. In section 2, we discuss MI in the nonlocal optical Kerr media with a sine-oscillatory response function in the framework of NNLSE [29], and it is shown that MI in this system has two unique properties that have not been reported before. In section 3, we discuss the physical mechanism behind MI in both local and nonlocal optical Kerr media by utilizing the theory of FWM. Although the link between MI and the FWM process has already been established for a local case [7,11,12], this work is the first exploration of the physical mechanism behind MI by extending the FWM method from the local case into the nonlocal one. The necessary and sufficient condition for MI to occur, which explains whether MI occurs in the optical Kerr media for different cases, is established in this work.

## 2. MI in Kerr media with a sine-oscillatory response

In the 1+1 dimensional lossless nonlocal optical Kerr media, a linearly polarized monochromatic optical beam propagating along the $z$ axis can be described by the nonlocal nonlinear Schrödinger equation (NNLSE) [13,20,29]

$$i\frac{\partial U}{\partial z} + \frac{1}{2k}\frac{\partial^2 U}{\partial x^2} + \frac{n_2 k}{n_0}U\int_{-\infty}^{\infty} R(x-x')|U(x',z)|^2 dx' = 0, \tag{4}$$

where $U$ is the slowly varying complex amplitude of the electric field satisfying the relation $E = U\exp(ikz)$, $k$ is the wavenumber in the media with a linear refractive index ($n_0$), and the Kerr coefficient $n_2$ can be either positive or negative. For different nonlocal optical Kerr systems, the response functions ($R(x)$) are different, such as the Gaussian function [29], exponential-decay function [17], rectangular function [29], and sine-oscillatory function [26,27].

The NNLSE has a plane wave solution

$$\overline{U} = I_0^{1/2}\exp\left[i\frac{2\pi n_2 k I_0 \widetilde{R}(0)}{n_0}z\right], \tag{5}$$

where $I_0$ is its intensity, and $\widetilde{R}(k_x)$ represents the Fourier transform of the response function

$$\widetilde{R}(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(x)\exp(-ik_x x)dx. \tag{6}$$

Then, we add a random perturbation on the plane wave such that

$$U(x,z) = [I_0^{1/2} + \psi(x,z)]\exp\left[i\frac{2\pi n_2 k I_0 \widetilde{R}(0)}{n_0}z\right], \tag{7}$$

to examine the evolution of the perturbation ($\psi(x,z)$) using linear stability analysis. Substituting Eq. (7) into Eq. (4) and linearizing with respect to $\psi$ as $|\psi(x,z)|^2 \ll I_0$, we obtain the linearized evolution equation of the perturbation as

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2k}\frac{\partial^2\psi}{\partial x^2} + \frac{2n_2 k I_0}{n_0}\int_{-\infty}^{\infty} R(x-x')\mathrm{Re}[\psi(x',z)]dx' = 0. \tag{8}$$

By decomposing the perturbation into real and imaginary parts ($\psi = u + iv$), and using the Fourier transform shown in Eq. (6), we derive a set of ordinary differential equations in the $k_x$ domain from Eq. (8)

$$\frac{d^2\widetilde{u}}{dz^2} + k_x^2\left[\frac{k_x^2}{4k^2} - \frac{2\pi n_2 I_0 \widetilde{R}(k_x)}{n_0}\right]\widetilde{u} = 0, \tag{9}$$

$$\frac{d^2\widetilde{v}}{dz^2} + k_x^2\left[\frac{k_x^2}{4k^2} - \frac{2\pi n_2 I_0 \widetilde{R}(k_x)}{n_0}\right]\widetilde{v} = 0. \tag{10}$$

By solving Eqs. (9) and (10),

$$\widetilde{\psi}(k_x,z) = \widetilde{u} + i\widetilde{v} = c_1\exp(\lambda z) + c_2\exp(-\lambda z) \tag{11}$$

is obtained, where $c_1$ and $c_2$ are arbitrary constants, and the eigenvalue $\lambda$ is given by

$$\lambda = |k_x|\left[\frac{2\pi n_2 I_0 \widetilde{R}(k_x)}{n_0} - \frac{k_x^2}{4k^2}\right]^{1/2}. \tag{12}$$

It should be noted that if $\lambda$ is real, the spatial spectrum component of the perturbation ($\widetilde{\psi}$) grows exponentially with $z$; therefore, the nonlinear system has MI. By contrast, when $\lambda$ is imaginary, the plane wave is stable.

Now, we consider the case where the nonlocal response function is of the sine-oscillatory type (given by Eq. (3)) and the Fourier transform of the response function is [31]

$$\widetilde{R}(k_x) = \frac{1}{2\pi(1 - w_m^2 k_x^2)}. \tag{13}$$

The gain coefficient defined by $g = 2\mathrm{Re}[\lambda]$ can be obtained as

$$g = \mathrm{Re}\left\{2|k_x|\left[\frac{n_2 I_0}{n_0(1 - w_m^2 k_x^2)} - \frac{k_x^2}{4k^2}\right]^{1/2}\right\}, \tag{14}$$

which only exists when $n_2 I_0/n_0(1 - w_m^2 k_x^2) > k_x^2/4k^2$. As the gain coefficient is an even function of $k_x$, we discuss only the variation of g with $|k_x|$.

When $n_2 < 0$, MI occurs when

$$1 < w_m^2 k_x^2 < \frac{1}{2}\left(1 - \frac{16 w_m^2 k^2 n_2 I_0}{n_0}\right)^{1/2} + \frac{1}{2}. \tag{15}$$

Fig. 1 shows the gain spectra of MI for three values of the plane wave intensity. It can be seen that when $I_0$ increases, the region of MI expands and the gain coefficient at the same wavenumber also increases. This shows that the intensity of the plane wave tends to boost MI in this case. An interesting characteristic of the gain spectra is that the maximum gain points do not shift with the intensity of the plane wave. Instead, they are fixed at the points where $|k_x| = 1/w_m$. In addition, because the Kerr coefficient is negative ($n_2 < 0$), the MI gain bands only appear in the region where the spectral function (13) has negative values.