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Measurement of the velocity of a quantum object: A role of phase and group velocities

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ABSTRACT

We consider the motion of a quantum particle in a free space. Introducing an explicit measurement procedure for velocity, we demonstrate that the measured velocity is related to the group and phase velocities of the corresponding matter waves. We show that for long distances the measured velocity coincides with the matter wave group velocity. We discuss the possibilities to demonstrate these effects for the optical pulses in coherently driven media or for radiation propagating in waveguides.

1. Introduction

The fact that propagation of waves depends on dispersion of the media has been known for a longtime ago [1,2]. In particular, the speed of light differs from its value in a vacuum [3] due to the index of refraction ($V_{ph} = V_g = c/n$, where n is the index of refraction). Due to dispersion of the index of refraction, the phase and group velocity are different ($V_{ph} = c/n \neq V_g = c/(n + \omega\partial n(\omega)/\partial\omega)$). Quantum coherence effects, such as coherent population trapping (CPT) [4] and electromagnetically induced transparency (EIT) [5–8], have been the focus of a broad range of research activity for the past two decades since they drastically change the optical properties of the media. In EIT, for example, absorption practically vanishes in both the CW and the pulsed regime [6–10]. A medium with an excited quantum coherence, phaseonium [5], can be used to make an ultra-dispersive prism [11] which will have several orders of magnitude greater angular spectral dispersion compared to a conventional one. Also the bending of light has been demonstrated using a transverse dragging effect [12]. The corresponding steep dispersion results in the ultraslow (or ultrafast) propagation of light pulses [13–17]. This in turn will produce huge optical delays [17] and therefore ultrahigh enhancement in absolute and relative rotation sensing can be achieved [18].

During last years [3,19], the dispersion properties of light propagating in the media with strong dispersion properties have been in a focus of broad study. Properties of matter waves is related to the understanding of the behavior of the quantum objects. Contrary to optical waves, which do not have any dispersion in vacuum, the matter waves have very strong dispersion to be taken into account. Here, we are going to discuss the process of measurement of velocity of a quantum object, and the role of phase and group velocities of matter waves on the procedure. Recently, the topic has attracted a lot of attention because of reports in the press on the detection of neutrino's traveling faster than the light in vacuum [20]. The results were very controversial that stimulated broad discussions about the details of experiments [21].

In this paper, we consider the propagation of the matter waves for the case of nonrelativistic as well as relativistic quantum mechanics. But let us note that, in principle, the results are applicable to any situation where the wave picture can be involved. We show that the propagation properties of matter waves, namely, the phase and group velocities are important to determine the measured velocity of the particle.

2. Velocity measurement

In classical physics, the measurement of velocity of free particle is a relatively simple procedure. One should determine the initial position z_0 and the position of the object z at the time t. Then, the velocity is given by

$$V = \frac{z - z_0}{t}, \tag{1}$$

where V is the measured velocity of the object. It is assumed that it is possible to repeat the experiment as many times as needed to obtain the desired accuracy for the velocity.

In quantum physics, we also assume that we can repeat experiment

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as many time as needed to decrease experimental errors. But for the quantum case, the experimental errors are also related to the fundamental postulates of quantum mechanics. In particular, to the fact that there is an initial spread of the particle position that is related to the initial width of a wave function. Thus, at time t=0, we have the initial wave function $\psi_0(z)$, and later at time t, we have the wave function $\psi(z, t)$. Then the measured velocity V is given by

$$V = \frac{\langle\psi|z|\psi\rangle - \langle\psi_0|z|\psi_0\rangle}{t}, \tag{2}$$

where

$$\langle\psi|z|\psi\rangle = \int_{-\infty}^{+\infty} dz\,|\psi(z,t)|^2 z, \text{ and} \tag{3}$$

$$\langle\psi_0|z|\psi_0\rangle = \int_{-\infty}^{+\infty} dz\,|\psi_0(z)|^2 z. \tag{4}$$

3. The matter waves and their dispersion: Schrodinger and Dirac equations

We consider the wave function describing the behavior of the matter waves. For the case of nonrelativistic motion, the wave function is determined by the Schrodinger equation that can be written as

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi. \tag{5}$$

Here, Eq. (5) is written in free space. Let us consider the wave function in the form of the plane wave

$$\Psi = A\exp[ikz - i\omega t] \tag{6}$$

being a solution of Eq. (5), where A is the amplitude of the matter wave. The energy and momentum can be related to the frequency $\omega$and wavenumber k of matter wave as $E = \hbar\omega$ and $p = \hbar k$. Then, the energy is related to the momentum as

$$E = \hbar\omega = \frac{\hbar^2k^2}{2m}, \tag{7}$$

establishing the dispersion of the matter waves. Then, the group velocity is given by

$$V_g = \frac{\hbar k}{m} = V_{cl} \tag{8}$$

which coincides with velocity of a classical motion of a particle V_{cl}, and the phase velocity is

$$V_{ph} = \frac{\hbar k}{2m} = \frac{V_{cl}}{2}. \tag{9}$$

For the case of relativistic motion of a particle, the wave function

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix},$$

is a spinor, and Ψ obeys the Dirac equation that is given by

$$i\hbar\partial_t\Psi = (c\vec{\alpha}\hat{p} + \beta mc^2)\Psi \tag{10}$$

where

$$\vec{\alpha} = \begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}, \text{ and} \beta = \begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},$$

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_y = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \text{ and} \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Looking for a plane wave solution of Eq. (10) as

$$\Psi = \exp(ikz - i\omega t)\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix},$$

$$i\hbar\partial_t\Psi = \hbar\omega\exp(ikz - i\omega t)\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}, \tag{11}$$

$$(c\vec{\alpha}\hat{p} + \beta mc^2)|\Psi\rangle = (c\vec{\alpha}\hat{p} + \beta mc^2)\exp(ikz - i\omega t)\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \tag{12}$$

$$\left[\hbar kc\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix} + mc^2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\right]\exp(ikz - i\omega t)\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}, \tag{13}$$

$$\hbar\omega\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \left[\hbar kc\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix} + mc^2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\right]\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}, \tag{14}$$

Then, the condition for nontrivial solution is given by

$$\begin{pmatrix}mc^2 - \hbar\omega & 0 & \hbar kc & 0\\ 0 & mc^2 - \hbar\omega & 0 & -\hbar kc\\ \hbar kc & 0 & -mc^2 - \hbar\omega & 0\\ 0 & -\hbar kc & 0 & -mc^2 - \hbar\omega\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = 0, \tag{15}$$

Then, the energy is given by

$$\hbar\omega = \sqrt{\hbar^2k^2c^2 + m^2c^4} \tag{16}$$

and corresponding the phase and group velocities are given by

$$V_g = \frac{c}{\sqrt{1 + \frac{m^2c^2}{\hbar^2k^2}}} \text{and} V_{ph} = c\sqrt{1 + \frac{m^2c^2}{\hbar^2k^2}}. \tag{17}$$

4. Measurement of particle velocity: the role of phase and group velocities of matter waves

Let us reconsider Eq. (1) in terms of the phase and group velocities of the matter waves. Now, we assume that the initial position of a particle has the Gaussian distribution with some initial momentum, and, then, knowing the dispersion of the matter wave, we can analyze the behavior of the matter waves later in time.

First, we assume that the initial (at t=0) wave function is

$$\psi(z, 0) = A\exp\left(-\frac{z^2}{2a^2} + ik_0z\right) \tag{18}$$

where $A = 1/\sqrt{\pi}a$ is the normalization constant. Then, we can write the initial wave function as

$$\psi(z, 0) = A\exp\left(-\frac{z^2}{2a^2} + ik_0z\right) = A\int_{-\infty}^{+\infty} dk\exp\left(-\frac{a^2(k-k_0)^2}{2} + ikz\right). \tag{19}$$

Note here that

$$\omega = \frac{\hbar k^2}{2m} \text{and} \omega_0 = \frac{\hbar k_0^2}{2m}, \tag{20}$$

and it is useful to write

$$\omega = \frac{\hbar k^2}{2m} = \frac{\hbar(k - k_0 + k_0)^2}{2m} = \frac{\hbar(k-k_0)^2}{2m} + \frac{\hbar k_0(k-k_0)}{m} + \frac{\hbar k_0^2}{2m}. \tag{21}$$

Thus, we obtain