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Classical and quantum harmonic oscillators with time dependent mass and frequency: A new class of exactly solvable model

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A R T I C L E I N F O

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A B S T R A C T

The classical harmonic oscillator with time dependent mass and frequency is investigated to obtain a closed form exact analytical solution. It is found that the closed form analytical solutions are indeed possible if the time dependent mass of the oscillator is inversely proportional to the time dependent frequency. The scaled wronskian obtained from the linearly independent solutions of the equation of motion of the classical oscillator is used to obtain the solution corresponding to its quantum mechanical counterpart. The analytical solution of the present oscillator is used to obtain the squeezing effects of the input coherent light. In addition to the possibilities of getting the squeezed states, the present solution will be of use for investigating various quantum statistical properties of the radiation fields. As an example, we investigate the antibunching of the input thermal (chaotic) light coupled to the oscillator. Therefore, the appearance of the photon antibunching does not warrant the squeezing and vice-versa. The exact solution is obtained at the cost of the stringent condition where the product of time dependent mass and frequency of the oscillator is time invariant.

1. Introduction

In order to explain the basic physics, the model of a simple harmonic oscillator (SHO) plays an important role. It is because the model of SHO gives exact solution both in classical and in quantum mechanical pictures. The incorporation of dissipation and the anharmonicity are very often necessary to take care the physical situations correctly. It is difficult and sometime it is impossible to get the closed form analytical solution of the oscillator subject to damping and or anharmonicity. We do not have any problems with the dissipative/ nonconservative systems, as long as we are lying in the domain of classical physics. The classical mechanics is sufficiently robust to take care the dissipation. However, the dissipation is a big problem in quantum mechanics where the Hamiltonian depends on time. The straight forward canonical quantization to the problem of damped oscillator does not hold good. It is because the fundamental commutation relation for the operators goes to zero as the time t becomes infinity. In other words, the solution to the damped quantum oscillator in Heisenberg and in Schroedinger pictures require some special attentions. Lot of works have already been done to quantize the dissipative systems [1–10]. Note that the harmonic oscillator with time dependent mass and constant frequency may lead to the problem of a damped harmonic oscillator [2–4]. By using the time invariance of Wronskian, the quantization of the oscillator with time dependent mass and frequency are available in these publication [2–4]. Of course, lot of people have investigated these problems by using various methods. A recent review on this subject is devoted to take care these methods and their consequences [10].

Apart from the mathematical interest, the quantum harmonic oscillator with time dependent mass and/or time dependent frequency have significant roles in many branches of physics. Colegarve et.al [11] showed that the problem of a Febry-Perot cavity in contact with a heat reservoir lead to the problem of harmonic oscillator with time dependent mass and constant frequency. The harmonic oscillator with time dependent frequency describes the quantum motion of the particle in a Paul trap [12] and the corresponding equation of motion could be described by the well known Mathieu equation (Appendix A). In the studies of expanding universe, Lemos and Natividate [13] have taken care the problem of a harmonic oscillator with time dependent frequency and constant mass. In addition to the above examples, the quantum harmonic oscillator with time dependent mass and /or time dependent frequency are extensively investigated for the solutions and their possible applications in various quantum optical phenomena. For example, the coherent state for a harmonic oscillator with time dependent frequency is investigated by Moya-Cessa et.al [14–16]. By using the invariant operator method [17], Yeon et.al [18] obtained the squeezing operators for a quantum oscillator with time dependent frequency. By using the well known Lewis and Reisenfeld [19] invariant operator approach and by using some time dependent transformation, the coherent state for a harmonic oscillator with time dependent mass and frequency and a perturbative potential has also been taken care by Dantas et.al [20]. The possibilities of getting squeezed state from the input coherent state coupled to a quantum oscillator with time dependent mass and frequency are investigated [4]. Of late, Ciftja

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has derived the exact wavefunction of a harmonic oscillator with time dependent mass and frequency [21]. The possibilities of getting squeezed states from the quantum *driven harmonic oscillator of time dependent mass and frequency* (DHTDMF) are studied by using the number state basis [22] and by using coherent state basis [3]. It is also interesting to note that the propagator for a quantum DHTDMF has also been derived [23]. Of late, we establish that the model of a quantum DHTDMF arises when a strong pump interact with the second order nonlinear medium of very weak absorption [3]. It is also identified that the Hamiltonian of a quantum DHTDMF coincides exactly with that of the Two-photon Hamiltonian [24].

The present communication is put in the following manner. The next section deals with the Hamiltonian of a classical oscillator with time dependent mass and frequency. The solution of the quantum mechanical counterpart is obtained by using the scaled Wronskian defined by two linearly independent solutions. Of course, the solutions and the corresponding dynamics of the quantum oscillator are always unitary. In spite of the several investigation in the subject of oscillator with time dependent mass and frequency, we, however, do not find any attempts where the exact analytical solution is explored. Therefore, the present exact solution of a harmonic oscillator with time dependent mass and frequency is a fresh one and will be of help for the further advancement of the subject. The squeezing of the input coherent light coupled to the oscillator is examined in Section 2.1. The possibility of an exact solution (in terms of the integral involving the time dependent frequency) for the oscillator with time dependent mass and frequency is obtained when the product of mass and frequency is time independent. The proposed solution is tested for a physical system. The short time behavior of the time dependent frequency is found useful for getting the closed form solutions up to the desired orders in time.

2. The model hamiltonian

The Hamiltonian of a classical simple harmonic oscillator of time dependent mass $M(t)$ and frequency $\Omega(t)$ is given by

$$H = \frac{p^2}{2M(t)} + \frac{1}{2}M(t)\Omega^2(t)q^2 \tag{1}$$

The corresponding equations of motion assumes the following form

$$\dot{q} = \frac{p}{M(t)}\dot{p} = -M(t)\Omega^2(t)q \tag{2}$$

Therefore, $q(t)$ and $p(t)$ are the classical canonical position and momentum respectively. Finally, we are end up with the following homogeneous second order linear differential equation with variable coefficients

$$\ddot{q} + \frac{\dot{M}(t)}{M(t)}\dot{q} + \Omega^2(t)q = 0. \tag{3}$$

Interestingly, by using a small transformation, the above differential equation could be obtained from the Mathieu differential equation (see Appendix A). Now, for $M(t) = M(0)\exp(bt)$ and $\Omega(t) = \Omega(0)$, the Eq. (3) reduces to the equation of motion for a damped harmonic oscillator of constant frequency with damping constant b. For constant mass $M(t) = M(0)$, the Eq. (3) corresponds the equation of motion of a SHO of time varying frequency. It is clear that the solution of the equation (3) assumes the following form

$$q(t) = A_1(t)q(0) + A_2(t)\dot{q}(0) \tag{4}$$

where the parameters $A_1(t)$ and $A_2(t)$ depend on time. The canonically conjugate momentum is obtained from the knowledge of the position $x(t)$ (Eq. (4)) involving the first one of the Eq. (2). Now, The parameters $A_i(t)$ are given by

$$A_1(t) = \frac{q_1(t)\dot{q}_2(0) - q_2(t)\dot{q}_1(0)}{q_1(0)\dot{q}_2(0) - q_2(0)\dot{q}_1(0)} \tag{5}$$

$$A_2(t) = \frac{q_1(0)q_2(t) - q_1(t)q_2(0)}{q_1(0)\dot{q}_2(0) - q_2(0)\dot{q}_1(0)} \tag{6}$$

where $q_1(t)$ and $q_2(t)$ are two linearly independent solutions of the Eq. (3). It is clear that $A_1(0) = 1$, $\dot{A}_1(0) = 0$, $A_2(0) = 0$ and $\dot{A}_2(0) = 1$. Now, the scaled Wronskian corresponding to the differential Eq. (3) $W_s(t) = M(t)[q_1(t)\dot{q}_2(t) - q_2(t)\dot{q}_1(t)]$ is independent of time since $\frac{dw_s}{dt} = 0$. The solution (4) may be used to obtain the solution of the quantum mechanical counterpart of the classical oscillator governed by the Eq. (3). These may be achieved provided the classical variables (position $q(0)$ and momentum $\dot{q}(0)$) in Eq. (4) are replaced by their corresponding operators along with the fundamental commutation relation. Note that the time dependent variables A_i are to be taken as *c-numbers* since q_1 and q_2 are the solutions of a *c-number* differential equation. It is because, these variables depend on the solutions of c-number differential equations. Therefore, the solution for the quantum mechanical oscillator follows immediately as

$$\hat{q}(t) = A_1(t)\hat{q}(0) + \frac{A_2(t)}{M(0)}\hat{p}(0) \tag{7}$$

where the momentum operator $\hat{p}(t) = M(t)\dot{\hat{q}}$. Now we calculate the fundamental commutation relation for the position and momentum operators

$$[\hat{q}(t), \hat{p}(t)] = \frac{W_s(t)}{W_s(0)}[\hat{q}(0), \hat{p}(0)] = i. \tag{8}$$

where $\hbar = 1$ is assumed. The validity of the above fundamental commutation (8) ensures that the solution of the classical damped harmonic oscillator is valid for the solution of its quantum mechanical counterpart when the classical position and classical momentum are replaced by their corresponding operators. The invariance of the scaled Wronskian does the job of quantization of the harmonic oscillator with time dependent mass and frequency. The availability of the quantum mechanical solution governed by Eq. (7) and its canonically conjugate operators $\hat{p}$ may be used for the calculation of second order variances in terms of the initial states of the radiation fields.

2.1. Squeezing of input coherent light

In order to have some glimpse on the possibilities of the production of squeezed states, we define the position and momentum operators in terms of the usual dimensionless annihilation (a) and creation ($a^\dagger$) operators. Therefore, we have

$$\hat{q}(t) = \frac{1}{\sqrt{2M(t)\Omega(t)}}(a(t) + a^\dagger(t))\hat{p}(t) = -i\sqrt{\frac{M(t)\Omega(t)}{2}}(a(t) - a^\dagger(t)) \tag{9}$$

where $\hbar = 1$ is used. The annihilation operator at t=0 (i.e $a(0)$) imposes the condition on the ground state (|0>) of the harmonic oscillator with $a(0)|0> = 0$. By using Eq. (8), it is easy to check that $[a(t), a^\dagger(t)] = 1$. Now, by using Eq. (9), it is possible to express the quantized version of the classical Hamiltonian (1). Hence, We have

$$H_{quantize} = \Omega\left\{a^\dagger(t)a(t) + \frac{1}{2}\right\} \tag{10}$$

Now, We define the dimensionless quadrature operators $X(t) = \sqrt{M(t)\Omega(t)}\hat{q}(t)$ and $P(t) = \frac{1}{\sqrt{M(t)\Omega(t)}}\hat{p}(t)$. Now, these dimensionless quadrature operators $X(t)$ and $P(t)$ are expressed in terms of the annihilation and creation operators at t=0. Therefore, we have