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Boundary conditions and formation of pure spin currents in magnetic field

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A B S T R A C T

Schrödinger equation for an electron confined to a two-dimensional strip is considered in the presence of homogeneous orthogonal magnetic field. Since the system has edges, the eigenvalue problem is supplied by the boundary conditions (BC) aimed in preventing the leakage of matter away across the edges. In the case of spinless electrons the Dirichlet and Neumann BC are considered. The Dirichlet BC result in the existence of charge carrying edge states. For the Neumann BC each separate edge comprises two counterflow sub-currents which precisely cancel out each other provided the system is populated by electrons up to certain Fermi level. Cancelation of electric current is a good starting point for developing the spin-effects. In this scope we reconsider the problem for a spinning electron with Rashba coupling. The Neumann BC are replaced by Robin BC. Again, the two counterflow electric sub-currents cancel out each other for a separate edge, while the spin current survives thus modeling what is known as pure spin current – spin flow without charge flow.

1. Introduction

The standard notion of electric current implies the directional flow of electrons with no preferred spin orientation. This results into the charge current with vanishing net spin flow. If electron spins are correlated for certain reasons, then alongside with the electric current one observes what is known as the spin current [1–3]. Considerable amount of studies [4–14] are devoted to the issue of the pure spin current – the flow of electron spin without flow of electric charge.

Schrödinger equation for an electron confined to a two-dimensional strip is considered in the presence of homogeneous orthogonal magnetic field. It is shown that in the case of spinning electrons with Rashba spin-orbit interaction, the Robin boundary conditions (BC) imposed on the wave function along the edges produce pure spin currents. For the sake of clarity we start with spinless electrons in Section 2 and point out the difference between the dispersion relations produced by the Dirichlet and Neumann BC. In Section 3 we discuss the electric currents carried by edge states and show that for Neumann BC each of the two edges accommodates two counterflow electric currents which precisely cancel out each other, i.e. the electric conductance of a separate edge is zero. In Section 4 we reconsider the problem for spinning electrons with Rashba spin-orbit interaction. In that case the Neumann BC are replaced by the Robin BC, leading to the same conclusion regarding the precise cancellation of electric currents at each edge separately. In contrast, the spin current is found to be finite, meaning the occurrence of pure spin current.

2. Spinless electron in homogeneous orthogonal magnetic field

Quantum mechanical Hamiltonian is given by

$$H = \frac{1}{2m}(i\hbar\partial_n + eA_n)^2, \tag{1}$$

where A_n is the vector potential with $B = \partial_x A_y - \partial_y A_x$.

We study the system with the geometry of infinite length $-\infty < y < +\infty$ and finite width $x_L \le x \le x_R$ with $x_L = -\frac{1}{2}d$ and $x_R = +\frac{1}{2}d$. Correspondingly, solving the eigenvalue problem, the wave function $\psi(x, y)$ has to be exposed to some boundary conditions (BC) preventing the leakage of a matter across the edges.

The matter flow is described by matter currents

$$\mathcal{J}_n = \frac{1}{2im}[\psi^\dagger(\hbar\partial_n\psi - ieA_n\psi) - (\hbar\partial_n\psi - ieA_n\psi)^\dagger\psi]. \tag{2}$$

Then the BC imposed on $\psi(x, y)$ must guarantee vanishing of x-component of the current (2) at boundaries

$$\mathcal{J}_x(x_L, y) = \mathcal{J}_x(x_R, y) = 0. \tag{3}$$

These conditions can be realized in a different ways, and here we comment on the following two options. One is the Dirichlet BC

$$\psi(x_L, y) = \psi(x_R, y) = 0, \tag{4}$$

and the other one is the Neumann BC

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$$\partial_x \psi(x_L, y) = \partial_x \psi(x_R, y) = 0. \tag{5}$$

Both of these options reproduce (3), but lead to significantly distinct dispersion relations, hence to distinct physical outcomes. In order to make this statement clear we pass to solving the eigenvalue problem.

Usage of the Landau gauge $\boldsymbol{A} = (0, Bx)$ secures translational invariance of the Hamiltonian in y-direction. Then the wave function can be written as

$$\psi(x, y) = e^{+iky}\phi_k(\xi), \tag{6}$$

where k is the momentum, and $\xi \equiv \ell^{-1}x + k\ell$ with ℓ being the magnetic length set by ($eB < 0$ is assumed)

$$\frac{1}{\ell^2} = -\frac{eB}{\hbar}. \tag{7}$$

By use of (6) the aforementioned boundary conditions are reformulated in terms of $\phi_k(\xi)$ and appear as

$$\text{Dirichlet BC: } \phi_k(\xi_L) = \phi_k(\xi_R) = 0, \tag{8a}$$

$$\text{Neumann BC: } \phi'_k(\xi_L) = \phi'_k(\xi_R) = 0, \tag{8b}$$

where

$$\xi_L \equiv -\frac{1}{2}\ell^{-1}d + k\ell, \tag{9a}$$

$$\xi_R \equiv +\frac{1}{2}\ell^{-1}d + k\ell. \tag{9b}$$

The eigenvalue problem for H is reduced to the equation $\mathcal{H}_k\phi_k(\xi) = \epsilon(k)\phi_k(\xi)$ where

$$\mathcal{H}_k = -\frac{1}{2}\partial_\xi^2 + \frac{1}{2}\xi^2. \tag{10}$$

Parameterizing eigenvalues as $\epsilon = \nu + \frac{1}{2}$ the general solution appears as

$$\phi_k(\xi) = e^{-\frac{1}{2}\xi^2}\left[c_1 M\left(-\frac{1}{2}\nu, \frac{1}{2}, \xi^2\right) + c_2\xi M\left(\frac{1}{2} - \frac{1}{2}\nu, \frac{3}{2}, \xi^2\right)\right], \tag{11}$$

where $M(a, b, z)$ is the Kummer function, and the constants $c_{1,2}$ to be determined by boundary and normalization conditions.

Consider first the Dirichlet BC (8a). Using (11) these appear as

$$c_1 M\left(-\frac{1}{2}\nu, \frac{1}{2}, \xi_L^2\right) + c_2\xi_L M\left(\frac{1}{2} - \frac{1}{2}\nu, \frac{3}{2}, \xi_L^2\right) = 0, \tag{12a}$$

$$c_1 M\left(-\frac{1}{2}\nu, \frac{1}{2}, \xi_R^2\right) + c_2\xi_R M\left(\frac{1}{2} - \frac{1}{2}\nu, \frac{3}{2}, \xi_R^2\right) = 0. \tag{12b}$$

This system has nontrivial solution for $c_{1,2}$ only if the corresponding determinant vanishes. Employing the Kummer transformation $M(a, b, z) = e^z M(b - a, a, -z)$ this condition can be expressed as[1]

$$f_D(\xi_L) = f_D(\xi_R), \tag{13}$$

where

$$f_D(\xi) \equiv \frac{M(\frac{1}{2} + \frac{1}{2}\nu, \frac{1}{2}, -\xi^2)}{\xi M(1 + \frac{1}{2}\nu, \frac{3}{2}, -\xi^2)}. \tag{14}$$

Eq. (12) determines ν as a function of k, i.e. the dispersion $\epsilon(k) = \nu(k) + \frac{1}{2}$. Solving (13) numerically one obtains the dispersion law shown in the left panel of Fig. 1. It should be noted that the dispersion curves produced by Dirichlet BC have been discussed in [15].

Consider now the Neumann BC, which by use of (11) is brought to the form

$$f_N(\xi_L) = f_N(\xi_R), \tag{15}$$

where

$$f_N(\xi) \equiv \frac{\xi M(\frac{1+\nu}{2}, \frac{1}{2}, -\xi^2) + 2\nu\xi M(\frac{1+\nu}{2}, \frac{3}{2}, -\xi^2)}{(1 - \xi^2)M(\frac{2+\nu}{2}, \frac{3}{2}, -\xi^2) + \frac{2}{3}(1 - \nu)\xi^2 M(\frac{2+\nu}{2}, \frac{5}{2}, -\xi^2)}. \tag{16}$$

The corresponding curve takes the shape shown in the right panel of Fig. 1.

Dirichlet and Neumann BC produce similar flat segments in the energy curves. This feature reflects the flat structure of the standard Landau levels where $\epsilon'(k) = 0$. Distinction between the two BC arises around the segments with nontrivial dispersion: Neumann BC cause the occurrence of dips which are absent for Dirichlet BC. This observation is the main object of our interest.

Some remarks are in order before discussing the issue of aforementioned dips. Increasing the width d, the flat segments also become wider, while the dips acquire certain stable shape. For the sake of clarity we comment on the case of Neumann BC and consider the right dip ($k > 0$).

Introduce the quantity $\kappa \equiv k\ell - \frac{1}{2}d/\ell$ which measures the deviation of k from the value of $\frac{1}{2}d/\ell^2$. Then the condition (15) appears as

$$f_N(\kappa) = f_N(\kappa + d/\ell). \tag{17}$$

Provided we discuss the vicinity of $k = \frac{1}{2}d/\ell^2$ with d/ℓ being large, the value of κ is finite. Then the right hand side of (17) can be replaced by the corresponding limit, and we come to

$$f_N(\kappa) = -\frac{\Gamma(\frac{1}{2} - \frac{1}{2}\nu)}{\frac{1}{2}\Gamma(-\frac{1}{2}\nu)}. \tag{18}$$

This relation generates infinite solutions for $\nu(\kappa)$ corresponding to the dips at $k > 0$. Hereafter we discuss only the right dips since the identical analysis is valid for the left ones, as well.

Remark, that the wave functions with momenta from plateaux take nonvanishing values at $x_L \le x \le x_R$, i.e. are bulk states, while those from the dips ($k \sim \pm\frac{1}{2}\ell^{-2}d$), are localized at boundaries thus representing the edge states. In particular, the states with $k \sim +\frac{1}{2}\ell^{-2}d$ are localized at the left edge $x \sim -\frac{1}{2}d$, and those with $k \sim -\frac{1}{2}\ell^{-2}d$ at the right edge $x \sim +\frac{1}{2}d$ (certain explicit expressions are collected in Appendix).

3. Matter current

Translational invariance forces the wave functions to take the form (6), and the eigenvalue problem becomes one-dimensional on the segment $\xi_L \le \xi \le \xi_R$. Correspondingly, the scalar product of two wave functions is defined as

$$\langle\phi|\varphi\rangle = \int_{\xi_L}^{\xi_R} \phi^*(\xi)\varphi(\xi)d\xi. \tag{19}$$

Elementary calculations indicate that within the class of wave functions set either by Dirichlet or Neumann BC we have $\langle\phi|\mathcal{H}\varphi\rangle = \langle\mathcal{H}\phi|\varphi\rangle$ signifying that $\mathcal{H}$ is hermitian. Provided $\phi_k(\xi)$ is the normalized eigenfunction we have $\epsilon(k) = \langle\phi_k|\mathcal{H}\phi_k\rangle$ where from we obtain

$$\frac{d\epsilon}{dk} = \int_{\xi_L}^{\xi_R} \phi_k^*\frac{d\mathcal{H}}{dk}\phi_k d\xi + \int_{\xi_L}^{\xi_R} \frac{d\phi_k^*}{dk}\mathcal{H}\phi_k d\xi + \int_{\xi_L}^{\xi_R} \phi_k^*\mathcal{H}\frac{d\phi_k}{dk}d\xi. \tag{20}$$

Due to $\langle\phi|\mathcal{H}\varphi\rangle = \langle\mathcal{H}\phi|\varphi\rangle$ the last two terms cancel out each other and we find

$$\frac{d\epsilon}{dk} = \ell\int_{\xi_L}^{\xi_R} \xi\,\phi_k^*(\xi)\phi_k(\xi)d\xi. \tag{21}$$

[1] Reasoning for Kummer transformation: increasing ξ^2, the value of $M(a, b, \xi^2)$ becomes exponentially large, while $M(a, b, -\xi^2)$ does not, hence more appropriate for numeric calculations.