IET Control Theory & Applications

Brief Paper

Input-delay approach to sampled-data $\mathscr{H}_\infty$ control of polynomial systems based on a sum-of-square analysis

ISSN 1751-8644
Received on 5th August 2016
Revised 6th January 2017
Accepted on 16th January 2017
E-First on 10th March 2017
doi: 10.1049/iet-cta.2016.1037
www.ietdl.org

Han Sol Kim[1], Jin Bae Park[1] ✉, Young Hoon Joo[2]

[1]*School of Electrical and Electronic Engineering, Yonsei University, Seoul 120-749, Korea*

[2]*Department of Control and Robotics Engineering, Kunsan National University, Kunsan, Korea*

✉ *E-mail: jbpark@yonsei.ac.kr*

Abstract: In this study, the authors develop an $\mathscr{H}_\infty$ stabilisation condition for polynomial sampled-data control systems with respect to an external disturbance. Generally, continuous-time and sampled state variables are mixed in polynomial sampled-data control systems, which is the main drawback to numerically solving the stabilisation conditions of these control systems. To overcome this drawback, this study proposes novel stabilisation conditions that address the mixed-states problem by casting the mixed states as a time-varying uncertainty. The stabilisation conditions are derived from a newly proposed polynomial time-dependent Lyapunov–Krasovskii functional and are represented as a sum-of-squares, which can be solved using existing numerical solvers. Some additional slack variables are further introduced to relax the conservativeness of the authors' proposed approach. Finally, some simulation examples are provided to demonstrate the effectiveness of their approach.

1 Introduction

Recently, sum-of-squares (SOS)-based stability analysis and controller synthesis have received significant attention from researchers in the field of control engineering for treating non-linear systems represented in polynomial vector fields [1–8]. The dynamic behaviour of many practical non-linear systems can be expressed by a polynomial system model; examples include aircraft systems [9], unmanned aerial vehicles [10], Moore–Greitzer model of a jet engine [11], and chaotic systems [12]. In addition, SOS-based approaches provide a systematic framework for investigating the stability of non-linear control systems, and stability conditions can be determined numerically by using SOS solvers such as SOSTOOLS [13, 14]. While a number of studies have dealt with polynomial control systems, very few studies have addressed sampled-data control of polynomial systems. Herein, a control system is defined as a sampled-data control system in which the analogue system is controlled by a digital controller [15]. Although some research works addressed polynomial control systems in the discrete-time domain [7, 8], these works cannot be directly extended to the sampled-data control problem. The reason for this is that this approach requires a discretised model of a continuous-time polynomial model; however, it is not easy to discretise a continuous-time polynomial model prior to investigating its stability. Moreover, to the best of our knowledge, no systematic and exact method has yet been proposed for discretising continuous-time polynomial control systems in a polytopic structure [16].

The sampled-data control problem for both linear and non-linear systems alike falls into two categories: (i) the direct discretisation method [16–19] and (ii) the input-delay approach [20–24]. In the direct discretisation method, the stability of a sampled-data control system is directly analysed in the discrete-time domain based on the discretised model specified in advance. The exact discrete-time polytopic models exist for linear-time-invariant systems; however, this is not the case for polynomial systems. Therefore, a previous study [16] attempted to discretise given continuous-time systems by ignoring higher-order terms and compensating for truncated errors. Due to the stability is investigated in the discrete-time domain, this method discards a large number of existing continuous-time control theories, making the approach less attractive to researchers.

On the other hand, the input-delay approach transforms a given sampled-data control problem into an equivalent input-delay control problem, and it has been successfully adopted for dealing with sampled-data control systems. This approach appears to be an appropriate solution to the sampled-data control of polynomial systems; however, studies thus far have been unable to account for the problem of mixed states in which the stability condition includes a combination of continuous-time and sampled state variables, which makes the condition non-convex. Furthermore, in previous studies, stability conditions were derived using a simple Lyapunov–Krasovskii functional (LKF) that employs constant Lyapunov matrices, thereby yielding conservative stability conditions.

In view of the above-described motivations, this paper develops an SOS-based input-delay approach to the stabilisation of sampled-data polynomial control systems. The proposed approach effectively accounts for the problem of mixed states by casting the difference among these continuous and discrete time states as a time-varying uncertainty. Moreover, the stabilisation condition is further extended to cover the $\mathscr{H}_\infty$ control design criterion, making the control system robust with respect to an external disturbance. The stability conditions are derived from a newly proposed polynomial time-dependent LKF in which polynomial Lyapunov matrices are employed and formulated in terms of SOSs. Together with the polynomial Lyapunov matrices, additional slack variables further relax the condition. Finally, simulation examples are provided to demonstrate the effectiveness of the proposed approach.

The following are the main contributions of this paper:

i. The mixed states included in the stability condition are effectively manipulated in a novel way.
ii. A novel polynomial time-dependent LKF and some slack variables are introduced to relax the stability condition.
iii. Both asymptotic stabilisation and $\mathscr{H}_\infty$ control design are considered for polynomial systems.

Notations: Vectors $v(t)$, $v(t_k)$, and $v(\tau)$ are expressed as v_t, v_k, and v_τ, respectively. The kth element of a vector v_t is denoted by v_t^k. For a polynomial matrix $M(x_t)$, $M^k(x_t)$ denotes its kth row vector.

For a matrix X, expressions such as $\mathrm{He}\{X\} = X + X^{\mathrm{T}}$ are employed to simplify the equations.

2 Problem formulation

In this paper, we deal with a control system whose dynamic equation can be modelled as the following polynomial model:

$$\begin{cases} \dot{x}_t = A(x_t)x_t + B_u(x_t)u_t + B_w(x_t)w_t \\ y_t = C(x_t)x_t, \end{cases} \tag{1}$$

on a compact set $\mathscr{B}_x \times \mathscr{B}_u := \{x_t : \| x_t \| \le \Delta_x\} \times \{u_t : \| u_t \| \le \Delta_u\}$, where $x_t \in \mathbb{R}^n$ is a state vector, $u_t \in \mathbb{R}^m$ is an input vector, $w_t \in \mathscr{L}_2^l[0, \infty)$ is an external disturbance, $y_t \in \mathbb{R}^p$ is an output vector, $A(x_t) \in \mathbb{R}^{n \times n}$, $B_u(x_t) \in \mathbb{R}^{n \times m}$, $B_w(x_t) \in \mathbb{R}^{n \times l}$, and $C(x_t) \in \mathbb{R}^{p \times n}$ are polynomial matrices of state variables, and Δ_x and Δ_u are positive scalars.

In this paper, we employ the following polynomial sampled-data controller:

$$u_t = u_k = K(x_k)x_k, \quad \text{for } t \in [t_k,\ t_{k+1}), \tag{2}$$

where $K(x_k) \in \mathbb{R}^{m \times n}$ is a polynomial control gain matrix to be determined, and t_k with $k \in \mathbb{Z}_{>0}$ denotes the kth sampling time. Thus, the control signal holds a constant value u_k for $t \in [t_k, t_{k+1})$ in which u_k denotes a control signal at $t = t_k$.

By substituting the polynomial sampled-data controller (2) into the polynomial system (1), we obtain the following closed-loop system representation:

$$\begin{aligned} \dot{x}_t &= A(x_t)x_t + B_u(x_t)K(x_k)x_k + B_w(x_t)w_t \\ &= A(x_t)x_t + B_u(x_t)K(x_k)(x_k - x_t + x_t) + B_w(x_t)w_t \\ &= \big(A(x_t) + B_u(x_t)K(x_k)\big)x_t - (t - t_k)B_u(x_t)K(x_k)\bar{x}_t + B_w(x_t)w_t \\ &= \phi(x_t, x_k)x_t - (t - t_k)\bar{\phi}(x_t, x_k)\bar{x}_t + B_w(x_t)w_t, \end{aligned} \tag{3}$$

where $\phi(x_t, x_k) = A(x_t) + B_u(x_t)K(x_k)$, $\bar{\phi}(x_t, x_k) = B_u(x_t)K(x_k)$, and

$$\bar{x}_t = \frac{1}{t - t_k}\int_{t_k}^{t} \dot{x}_\tau \, \mathrm{d}\tau = \frac{1}{t - t_k}(x_t - x_k). \tag{4}$$

The following lemmas are required to derive the proposed stabilisation condition.

Lemma 1 [16]: Given the matrices U, V, and $\Gamma = \Gamma^{\mathrm{T}}$ of appropriate dimensions, the following statements are equivalent:

i. $\Gamma + \mathrm{He}\{U\Delta V\} \prec 0, \forall \Delta \in \{M : M^{\mathrm{T}}M \le \sigma^2 I\}$.
ii. There exists a real number $\kappa > 0$ such that $\Gamma + \kappa^{-1}V^{\mathrm{T}}V + \kappa\sigma^2 UU^{\mathrm{T}} < 0$ holds.
iii. There exists $\kappa \in \mathbb{R}$ such that the following condition holds:

$$\begin{bmatrix} \Gamma + \kappa\sigma^2 UU^{\mathrm{T}} & * \\ V & -\kappa I \end{bmatrix} \prec 0.$$

In this paper, all stability conditions are derived on the basis of the Lyapunov stability theory [25] and are represented in the form of SOSs, allowing us to solve the conditions numerically. The following lemma describes the relationship between the positivity of a polynomial matrix and an SOS condition.

Lemma 2 [14]: Any symmetric polynomial matrix $M(x_t) = M^{\mathrm{T}}(x_t)$ is positive definite if the following SOS condition is satisfied:

$$v^{\mathrm{T}}\big(M(x_t) - \epsilon(x_t)I\big)v \text{ is SOS}, \tag{5}$$

where $\epsilon(x_t) > 0$ for $x_t \ne 0$ is a predefined scalar polynomial function of x_t, and v is an arbitrary vector of an appropriate dimension that is independent of x_t.

In this paper, motivated by the work in [21], we propose the following polynomial time-dependent LKF for deriving the stabilisation condition of (3):

$$V(t) = V_1(t) + V_2(t), \quad t \in [t_k, t_{k+1}), \tag{6}$$

where

$$\begin{aligned} V_1(t) &= x_t^{\mathrm{T}}P(\tilde{x}_t)x_t, \\ V_2(t) &= (t_{k+1} - t)\int_{t_k}^{t} \dot{x}_\tau^{\mathrm{T}}R(\tilde{x}_t)\dot{x}_\tau \, \mathrm{d}\tau, \end{aligned}$$

$0 \prec P(\tilde{x}_t) = P^{\mathrm{T}}(\tilde{x}_t) \in \mathbb{R}^{n \times n}$, $0 \prec R(\tilde{x}_t) = R^{\mathrm{T}}(\tilde{x}_t) \in \mathbb{R}^{n \times n}$, $\tilde{x}_t := \begin{bmatrix} x_t^{k_1} & x_t^{k_2} & \dots & x_t^{k_q} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^q$, and $\mathscr{K} := \{k_1, k_2, \dots, k_q\}$ denotes the set of the row indices of both $B_u(x_t)$ and $B_w(x_t)$ whose corresponding rows are zero. For example, if

$$B_u(x_t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_w(x_t) = \begin{bmatrix} 1 + x_t^1 \\ 0 \\ 0 \end{bmatrix},$$

i.e. if the second and third rows are simultaneously zero, then we set

$$\mathscr{K} = \{2, 3\} \quad \text{and} \quad \tilde{x}_t = \begin{bmatrix} x_t^2 \\ x_t^3 \end{bmatrix}.$$

Remark 1: As $P(\tilde{x}_t)$ and $R(\tilde{x}_t)$ are positive definite polynomial matrices, $V_1(t)$ and $V_2(t)$ are also positive definite functions. Clearly, $V_1(t)$ is continuous in time and differentiable for all $t \in [0, \infty)$ and $V_2(t)$ is continuous on $t \in [0, \infty)$ because $\lim_{t \to t_k} V_2(t) = V_2(t_k) = 0$. Moreover, $V_2(t)$ is differentiable for $t \in (0, \infty)$ except for all $t = t_k$ with $k \in \mathbb{R}_{>0}$.

The purpose of this paper is to derive a sufficient condition for guaranteeing the stability of the closed-loop polynomial system (1) closed with the non-linear sampled-data polynomial controller (2) under $w_t = \mathbf{0}$. Moreover, we also deal with the $\mathscr{H}_\infty$ control problem so as to demonstrate the applicability of the proposed approach.

3 Asymptotic stabilisation of sampled-data polynomial control systems

In this section, the sampled-data stabilisation condition for the closed-loop polynomial system (3) is provided under $w_t = \mathbf{0}$. The following lemma introduces slack variables by which the condition is relaxed.

Lemma 3: The following inequality is satisfied:

$$\int_{t_k}^{t} \dot{x}_\tau^{\mathrm{T}}\big((t_{k+1} - t)\dot{R}(\tilde{x}_t) - R(\tilde{x}_t)\big)\dot{x}_\tau \, \mathrm{d}\tau \le (t - t_k)\bar{x}_t^{\mathrm{T}}\big(L(\tilde{x}_t) + 2H\big)\bar{x}_t, \quad \text{for } t \in [t_k, t_{k+1}) \tag{7}$$

If there exist positive definite matrices $0 \prec R(\tilde{x}_t) = R^{\mathrm{T}}(\tilde{x}_t)$ and $0 \prec L(\tilde{x}_t) = L^{\mathrm{T}}(\tilde{x}_t)$, and symmetric matrix $H = H^{\mathrm{T}}$ of appropriate dimensions such that the following matrix inequalities hold:

$$R(\tilde{x}_t) - T\dot{R}(\tilde{x}_t) \succ 0, \tag{8}$$