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Effective elastic constants of hexagonal array of soft fibers

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ABSTRACT

Analytical formulae for the effective elastic constants of 2D composites with soft unidirectional fibers arranged in the hexagonal array are obtained for arbitrary concentration of fibers, from dilute case to percolation. It is supposed that every section of composites perpendicular to fibers is the hexagonal array with circular holes or soft inclusions. First, a polynomial approximation in concentration is obtained by application of functional equations. Further, an asymptotic analysis is applied to the obtained polynomial with using of the known asymptotic formulae in percolation regime. Finally, the formula for the effective shear modulus is suggested. It can be applied to the typical matrices made from ceramics, metals and polymers, and to other materials as well, e.g., for resin and thallium.

1. Introduction

Accurate and consistent evaluation of the effective properties of composites and porous media is one of the fundamental tasks of applied mathematics and engineering. Experimental methods require advanced technological methods in order to obtain accurate results [1]. First of all this concerns the transverse shear modulus of fibrous composites. Therefore, theoretical investigations by means of analytical and numerical techniques are paramount. In particular, they are important for the regime with high-concentration of fibers. Solution to boundary value problems for a multiply connected domain in a class of periodic functions leads to theoretical evaluation of the effective properties. In the present paper, we restrict ourselves to plane strain elastic problems with circular holes that corresponds to fibrous composites with very soft fibers. Such elastic problems have application in poroelasticity, in particular, in biomechanics [2] in order to estimate a stress-strain state of reconstruction system bone-implant. Approximate and analytical formulae for porous media are applied in geophysics [3,4]. Applications of fiber composites in industry is outlined in [5]. Aluminium bricks with cylindrical holes were widespread in constructions of engines and it is still used in certain applications where it remains advantageous.

The general potential theory of mathematical physics yields methods of integral equation to numerically solve various boundary value problems. Integral equations for plane elastic problems were constructed by Muskhelishvili [6], first, extended to doubly periodic problems in [7] and developed in [8–10]. The obtained results were applied to computations of the effective properties of the elastic media. Integral equations are efficient for the numerical investigation of a non-dilute composites when interactions of inclusions have to be taken into account.

Approximate analytical formulae were recently obtained for bi-Laplace's equation which describes elastic materials [11] for a circular multiply connected domain. It is worth noting that many efforts were applied to get analytical formulae. The results were obtained for two extremal regimes when the concentration of inclusions f is low [12–15] and near the percolation threshold [12,15]. In the present paper, we apply the method of functional equations [16,17,11] to elastic plane problems for the regular hexagonal (triangular) array of holes. In this particular, but important in application case, we show how to deduce high order concentration formulae for general random 2D composites.

The second step of investigations is asymptotic analysis to establish analytic formula for the effective shear modulus near the percolation threshold and to obtain a universal formula valid for all f. For an array of circular holes on the cites the hexagonal lattice effective shear modulus is expected to decay in the vicinity of $f_c = \frac{\pi}{12} \approx 0.9069$ as a power-law,

$$\mu_e \simeq A(f_c - f)^{\mathcal{T}},$$

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with positive critical index $\mathcal{T}$. The phase-interchange theorem [15], does not hold in the case of shear modulus for such array of holes, i.e., it should be two different values for the critical index for holes and inclusions, but does hold for the bulk-modulus [15].

For the antiplane shear problem the elastic displacements are zero in the plane but are non-zero in the direction perpendicular to the plane, and in the isotropic case only scalar elastic shear modulus μ participates. The 2D dielectric constant (or electrical conductivity) problem is rigorously analogous to the antiplane shear elasticity problem [15,18], so that all results concerning effective properties discussed above can be applied to the effective shear modulus μ^* as a function of contrast parameter $\rho_\mu = \frac{\mu_1-\mu}{\mu_1+\mu}$, and concentration of the inclusions f. Here, μ_1 and μ stand for the shear modulus of the inclusions and matrix, respectively.

De Gennes [19] conjectured that in the vicinity of percolation threshold, such properties as conductance of a resistor network and the effective elastic modulus for one-component elastic displacement of point-like monomers, behave analogously. I.e., in antiplane shear an effective elastic modulus for perfectly rigid inclusions should diverge as

$$\mu_e^* \sim (f_c - f)^{-\mathcal{S}}$$

in the vicinity of the critical concentration f_c. According to [19], the critical index $\mathcal{S}$ for the elastic modulus is equal to the superconductivity index s. By analogy we expect $\mathcal{S} \simeq 1.3$ in random case, and $\mathcal{S} = \frac{1}{2}$ for regular lattice arrangements of inclusions. In case of plane strain elasticity such quantity as bulk modulus behaves similarly and with the same critical index at least in the regular case [15], and will be considered elsewhere.

Another interesting exact result is independence of the effective Young modulus of a 2D sheet containing circular holes on the Poisson coefficient of the matrix [20,21]. Although it does not hold for rigid or other inclusions, actual dependence of the Poisson ratio ν is rather weak.

The holes are punched in the matrix with different elastic properties. Plane strain elastic problem is considered for such composite and the effective elastic modulus is obtained in the form of power series in the inclusions concentration and elastic constants for holes and matrix.

For holes the shear modulus $\mu_1 = 0$ and the Poisson ratio $\nu_1 = 0$. We construct an expansion for the effective shear modulus available up to $O(f^{14})$ inclusively.

2. Method of functional equations for local fields

We begin our study with a finite number n of inclusions on the infinite plane. This number n is given in a symbolic form with an implicit purpose to pass to the limit $n \to \infty$ later. The shear modulus of inclusions is also arbitrary taken as μ_1 to pass to the limit $\mu_1 \to 0$ in the final formulae. Introduce the complex variable $z = x + iy$ where i denotes the imaginary unit. Let inclusions be disks $D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 1, 2, \ldots, n)$ in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Denote $\partial D_k := \{t \in \mathbb{C} : |t - a_k| = r\}$ where the curves ∂D_k are oriented in clockwise sense. Let D be the complement of $\cup_{k=1}^n (D_k \cup \partial D_k)$ to $\widehat{\mathbb{C}}$ (see Fig. 1).

Let the uniform shear stress are applied at infinity

$$\sigma_{xx}^\infty = \sigma_{yy}^\infty = 0, \quad \sigma_{xy}^\infty = \sigma_{yx}^\infty = 1. \tag{1}$$

The component of the stress tensor can be determined by the Kolosov-Muskhelishvili formulae [6]

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\ \varphi_k'(z), & z \in D_k, \\ 4\mathrm{Re}\ \varphi'(z), & z \in D, \end{cases} \tag{2}$$

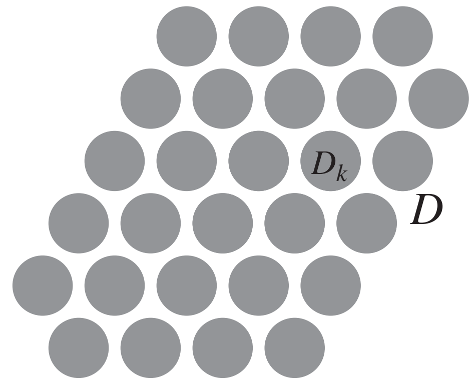

Fig. 1. Section of fibrous composite.

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], & z \in D_k, \\ -2\left[z\overline{\varphi''(z)} + \overline{\psi_0'(z)}\right], & z \in D, \end{cases} \tag{3}$$

where Re denotes the real part and the bar the complex conjugation. The component of the deformation tensor have the form

$$\epsilon_{xx} + \epsilon_{yy} = \begin{cases} \frac{\kappa_1 - 1}{\mu_1}\mathrm{Re}\,\varphi_k'(z), & z \in D_k, \\ \frac{\kappa - 1}{\mu}\mathrm{Re}\,\varphi'(z), & z \in D, \end{cases} \tag{4}$$

$$\epsilon_{xx} - \epsilon_{yy} + 2i\epsilon_{xy} = \begin{cases} -\frac{1}{\mu_1}\mathrm{Re}\left(\bar{z}\varphi_k''(z) + \psi_k'(z)\right), & z \in D_k, \\ -\frac{1}{\mu}\mathrm{Re}\left(\bar{z}\varphi''(z) + \psi_0'(z)\right), & z \in D, \end{cases} \tag{5}$$

where $\kappa = \frac{3-\nu}{1+\nu}$, ν is the 2D Poisson ratio [22]. The same notation is used for κ_1. We have $\psi_0(z) = iz + \psi(z)$. The functions $\varphi(z), \psi(z)$ are analytic in D, twice differentiable in $D \cup \partial D$ and bounded at infinity. The functions $\varphi_k(z)$ and $\psi_k(z)$ are analytic in D_k and twice differentiable in the closures of the considered domains.

The perfect bonding at the matrix-inclusion interface can be expressed by two conditions [6]

$$\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi_0(t)}, \tag{6}$$

$$\kappa_1\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)} = \frac{\mu_1}{\mu}\left(\kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi_0(t)}\right). \tag{7}$$

Introduce the new unknown functions

$$\Phi_k(z) = \left(\frac{r^2}{z - a_k} + \overline{a_k}\right)\varphi_k'(z) + \psi_k(z), \quad |z - a_k| \leqslant r,$$

analytic in D_k except the point a_k where its principal part has the form $r^2(z - a_k)^{-1}\varphi_k'(a_k)$. Introduce the inversion with respect to the circle ∂D_k

$$z_{(k)}^* = r^2\left(\overline{z - a_k}\right)^{-1} + a_k.$$

The problem (6), (7) was reduced in [16] (see Eqs. (5.6.11) and (5.6.16) in Chapter 5), [23] to the system of functional equations

$$\left(\frac{\mu_1}{\mu} + \kappa_1\right)\varphi_k(z) = \left(\frac{\mu_1}{\mu} - 1\right)\sum_{m \neq k}\left[\overline{\Phi_m(z_{(m)}^*)} - (z - a_m)\overline{\varphi_m'(a_m)}\right] - \left(\frac{\mu_1}{\mu} - 1\right)\overline{\varphi_k'(a_k)}(z - a_k) + p_0, |z - a_k| \leqslant r, \tag{8}$$