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An efficient center manifold technique for Hopf bifurcation of n-dimensional multi-parameter systems

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ABSTRACT

The center manifold theory with respect to the simple Hopf bifurcation of a n-dimensional nonlinear multi-parametric system is treated via a proper symbolic form. Analytical expressions of the involved quantities are obtained as functions of the parameters of the system via effective algorithms based on the followed procedure and carried out using a symbolic computation software. Moreover the normal form of a codimension 1 Hopf bifurcation, as well as the corresponding Lyapunov coefficient and bifurcation portrait, can be computed for any system under consideration. Here the computational procedure is applied to two nonlinear three-dimensional, three-parametric systems and graphical results are obtained as concerns the stability regions, the bifurcation portraits, as well as emerged limit cycles with respect to both the supercritical and the subcritical case of bifurcation.

1. Introduction

An extensive number of n-dimensional nonlinear dynamical systems ($n \geq 3$) in various fields of applied sciences are encountered in the literature. These are studied with respect to stability and local bifurcations caused by the variation of bifurcation parameters. There are examples in many scientific disciplines such as biosciences [1–4], energy systems [5,6], economics [7,8] etc. as well as in classic dynamical systems such as the Lorenz model [9], Lü [10,11] and the Chen [12] model. Regarding the Hopf bifurcation, the combination of the center manifold theory and the Poincaré normal forms for planar systems, applied to n-dimensional ones, is presented in the classic book of Kuznetsov [13]. The method is based on the state space decomposition in the critical and non-critical subspace in addition to the necessary normalization with respect to the involved eigenvectors. Then substitution of the reduced critical equations in the invariance relation of the center manifold (W^c) ([13], Sections 5.4.1, 8.7.1 and 8.7.3), leads to the computation of the critical polynomial coefficients of $W^c$and therefore to the evaluation of the coefficients of the equation restricted to the center manifold. Finally, according to the obtained two-dimensional normal forms ([13], Sections 3.5, 8.3), formulae for the *resonant* odd terms and the corresponding critical Lyapunov coefficients are obtained up to codimension 2 (*Bautin bifurcation*), and the corresponding bifurcation diagram is also shown.

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In our work, we focus on the simple (codimension 1) Hopf bifurcation. By adopting a symbolic representation, we obtain specification of the necessary order of the center manifold that should be kept in each term involved in the analytic calculations referring to the derivation of the restricted equation and the treatment of the invariance relation. We make extensive use of the symbolic algebra package Mathematica 7 in (Section 2, 2.2, 2.3).

The state space decomposition is applied here on the equilibrium path and hence the reduced equations, and all the formulae obtained, are expressed in terms of the system parameters. In particular the center manifold coefficients are derived as parameter functions (explicitly and implicitly via the eigen-quantities) and computed throughout the region of the parameter space of interest. Therefore the coefficients of the restricted equation expressed in terms of the eigenvectors and the center manifold coefficients and also the *resonant* term of the normal form (being function of the planar (restricted) coefficients), as well as the first Lyapunov coefficient, are numerically evaluated, not only at the critical parameter values, but throughout the whole parameter space. This allows the respective bifurcation portrait to be constructed. The parameter dependent formulae allow the computer assisted calculation of the derivatives of the various coefficients involved in the analysis and so allows the *transversality* condition to be verified at the critical equilibrium. The basic steps of the followed procedure are displayed in Section 2.3.

We note that as the analysis concerns a general multi-parameter system (we refer to the three-parameter case herein), in order to take into account the combined effects generated by the variation of as more as possible significant quantities of the physical background of the problem. We also note that the form and the structure of the representation adopted, as well as the obtained analytic formulae, allow the generalization and application of this procedure to any multi-dimensional and multi-parameter system, with minor modifications. Finally we would like to note that, as mentioned before, it is the structure and the steps of the analytical procedure that are shown here, while the formulae obtained, not listed here, are included in the respective electronic files, produced via algorithms based on the symbolic algebraic form of the analysis adopted herein. Relevant work regarding a resource efficient computational methodology for the evaluation of the normal form has been carried out by Tian and Yu [14], where a recursive methodology is used.

In Sections 3 and 4 we deal with two three-dimensional systems as applications of proposed symbolic computational procedure. These concern a modified Lorenz model and an autonomous Energy Resources model respectively. The stability analysis corresponding to each one of the systems, carried out in Sections 3.1 and 4.1, concludes with the appropriate parameters forming the three-dimensional parameter space of the bifurcation that the systems undergo. Then in Sections 3.2 and 4.2, stability regions and bifurcation portraits for each one of the systems are presented in the parameter planes of interest. Finally bifurcated limit cycles for both systems are obtained using a multiple shooting algorithm. The obtained cycles, valid for specific values of the involved parameters, concern both the supercritical and the subcritical case of bifurcation. In the subcritical case, the method proved particularly effective in the difficult task of obtaining an unstable cycle.

2. Analysis and formulae for the center manifold theory

2.1. Reduction to an ($\boldsymbol{n}+\boldsymbol{2}$)-dimensional coordinate space

Consider a smooth continuous-time three-parameter system with smooth dependence on the parameters:

$$\frac{dx}{dt} = f(x;a) \tag{2.1}$$

with $x \in \mathbb{R}^n$, $a = (a_1, a_2, a_3)^T \in \mathbb{R}^3$ and $f : \mathbb{R}^{n+3} \to \mathbb{R}^n$ with $f \in C^\infty$. If $x^0(a)$ is the equilibrium path of (2.1) and $J_0(a)=D_x f(x^0(a);\ a)$ the Jacobian matrix evaluated at this equilibrium path, then for small perturbations $\xi = x - x^0(a)$, expansion of (2.1) yields

$$\frac{d\xi}{dt} = J_0(a)\xi + F(\xi;a). \tag{2.2}$$

The smooth vector function $F : \mathbb{R}^{n+3} \to \mathbb{R}^n$ represents the nonlinear terms of the right-hand side of (2.2), that is $F = O(\|x\|^2)$ and

$$F(\xi;a) = \frac{1}{2}F^{(2)}(\xi,\xi;a) + \frac{1}{6}F^{(3)}(\xi,\xi,\xi;a) + \cdots, \tag{2.3}$$

where

$$F^{(m)}\left({}^1u,\ldots,{}^mu;a\right) = \sum_{j_1,\ldots,j_m=1}^{n} \left.\frac{\partial^m f(x;a)}{\partial x_{j_1}\ldots\partial x_{j_m}}\right|_{x=x^0} {}^1u_{j_1}\ldots{}^mu_{j_m}, \quad m = 2,3,\ldots \tag{2.4}$$

with ${}^1u,\ldots,{}^mu \in \mathbb{C}^n$. The Hopf bifurcation theorem (see [15], Section 11.2) implies that if at a critical triplet $a_0=(a_{10},a_{20},a_{30})$, the Jacobian matrix J_0 has a pair of purely imaginary eigenvalues $\lambda_{1,2}=\pm i\omega_0$, $\omega_0 > 0$, while the real part of the remaining eigenvalues is negative, then an oscillatory instability occurs close to a_0, leading to a family of limit cycles. Moreover in a small region around a_0, the Jacobian matrix J_0 has a pair of complex conjugate eigenvalues $\lambda(a),\ \bar{\lambda}(a)$ with

$$\lambda(a) = \mu(a) + i\omega(a), \quad \mu(a_0) = 0, \quad \omega(a_0) = \omega_0 > 0. \tag{2.5}$$