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A sixth-order finite volume method for diffusion problem with curved boundaries

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ABSTRACT

A sixth-order finite volume method is proposed to solve the Poisson equation for two- and three-dimensional geometries involving Dirichlet condition on curved boundary domains where a new technique is introduced to preserve the sixth-order approximation for non-polygonal or non-polyhedral domains. On the other hand, a specific polynomial reconstruction is used to provide accurate fluxes for elliptic operators even with discontinuous diffusion coefficients. Numerical tests covering a large panel of situations are addressed to assess the performances of the method.

1. Introduction

Very high-order finite volume method (higher than the second-order) for elliptic and parabolic operators on unstructured meshes is a recent trend and has received considerable attention during the last decade. The coupling of the Euler system with a viscous term [1–7], the incompressible Navier–Stokes equations [8–10] or the shallow-water system with turbulence [11] are, among others, strong motivations to design efficient and accurate schemes for elliptic operators in the finite volume context. There exists a large literature for the second-order approximations with convergence and stability analysis [12–19], but few studies have been done for very high-order approximations on unstructured meshes [7,9,10,20,21]. In a recent paper [22], a new method for convection–diffusion problems has been developed for two-dimensional geometries providing up to sixth-order approximations. The technique is based on specific polynomial reconstructions to evaluate the fluxes across the cell interfaces with a very high accuracy.

The goal of the present study is, on the one hand, to achieve an extension for the three-dimensional case and, on the other hand, to develop a new class of polynomial representation for the boundary to preserve an effective sixth-order approximation even with non-polygonal and non-polyhedral domains. Indeed, when dealing with second-order finite volume schemes for elliptic operator, the domain can be substituted by a polygonal or a polyhedral one and the Dirichlet condition is evaluated at the vertices or the midpoint of the boundary edges. The question of curved boundary is relevant for at least third-order schemes where the substitution with a polygonal or a polyhedral domain will reduce the global accuracy of the

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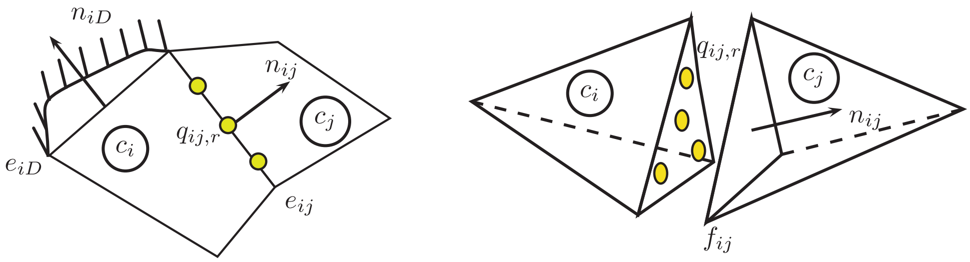

Fig. 1. Notation for the two-dimensional (left) and three-dimensional mesh.

scheme. In [3], the authors propose a polynomial reconstruction which exactly matches the Dirichlet condition at the Gauss points on the curved boundary. The procedure then involves two new linear constraints added to the matrix system we use to compute the polynomial coefficients and the method is extended to the Neumann condition. Another approach consists in applying a local mapping $(x, y) \to (\xi, \eta)$ which transforms the cell into a new computational domain to match with the boundary [23]. Integrals therefore involves the Jacobian matrix of the transformation and polynomial reconstruction procedure is performed in the local basis leading to a rather complex calculation. The method we propose in the present study is related to the works of [3] but the Dirichlet condition is enforced in a different way by using the mean value on the edge as a free parameter. Then, we determine the parameter such that the polynomial reconstruction and the Dirichlet condition corresponds on the curved arc or surface.

Another important issue concerns the approximation of the solution when dealing with different materials. Discontinuity of the coefficients takes place at the interface between two dielectrics and specific numerical schemes have to be designed to preserve the accuracy on both sides since the derivative of the solution may present a jump [24]. We have developed a technique for the second-order case [25] in the finite volume context and extension to the sixth-order is obtained in the present study.

Very high-order method leads to an important computational effort due to the reconstruction process and a large part of the time consumption derives from the polynomial coefficients evaluation. We proposed a new procedure to dramatically reduce the computational cost by calculating a local vector for each Gauss point one can identify as a partial assembly procedure. Such evaluation is carried out once during the pre-processing stage and it results that the polynomial evaluation at the Gauss points is reduced to a merely inner product between the local vector and the data, saving a lot of memory and time.

In the present document, we do not tackle the convective part on purpose since the main difficulty concerns the diffusive contribution so we only focus on the Poisson problem. The second section is devoted to the generic very high-order finite volume scheme while the third section deals with the polynomial reconstruction and detail all the improvements we propose. Numerical experiments are presented in the four section to assess the scheme accuracy and robustness as well as the efficiency of the iterative solver coupling with a new preconditioning strategy.

2. Generic high-order finite volume scheme

We consider an open bounded domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$ with a piecewise regular curved boundary. Γ_D and Γ_N define a partition of the boundary $\partial\Omega$ where we shall prescribe the Dirichlet and the Neumann conditions respectively. We intend to compute accurate approximations of function V solution of the Poisson equation

$$-\nabla.(\varepsilon\nabla V) = g, \quad \text{in } \Omega, \quad \varepsilon\nabla V.n = 0, \text{ on } \Gamma_N, \quad V = V_D \text{ on } \Gamma_D, \tag{1}$$

with ε a positive function which may present some discontinuities, V_D is a given function defined on Γ_D and g is the source term. We assume that boundary Γ_N is composed of lines (2D case) or planes (3D case) while Γ_D is curved. In some applications, we shall split the domain into two sub-domains Ω_1 and Ω_2 shared by an interface Γ and we denote by $\varepsilon_\ell = \varepsilon_{|\Omega_\ell}$, $\ell = 1, 2$ the restrictions on each sub-domain. Functions ε_1 and ε_2 are regular but ε presents a discontinuity at the interface Γ. Such an assumption is required when dealing with two materials with different physical characteristics which usually happen when dealing with several layers of dielectric or semiconductor [26].

We introduce the notations to derive the finite volume scheme (see Fig. 1). For the two-dimensional case, the domain is divided into non-overlapping convex polygonal cells c_i, $i = 1, \dots, I$ and

$$\Omega_h = \bigcup_{i=1}^{I} c_i$$

is the associated polygonal domain. We denote by $e_{ij} = c_i \cap c_j$ the common interface shared by two adjacent cells while n_{ij} stands for the unit normal vector from c_i towards c_j. To handle the edges on the boundary, we introduce the notation e_{iD} which corresponds to an edge of cell c_i which belongs to the boundary of the polygonal domain such that its vertices belong to Γ_D. Since $\Omega \neq \Omega_h$, we denote by $\Gamma_{h,D}$ the boundary of Ω_h constituted of the edges e_{iD} and vector n_{iD} is the unit outward normal vector of c_i on e_{iD}. In the same way we define e_{iN} and n_{iN} for the edge associated to Γ_N. At last ν_i