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# Dense underflow into a lake or reservoir – supercritical flow solutions

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A R T I C L E I N F O

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A B S T R A C T

Steady two-dimensional flow from an angled structure into a lake or a reservoir where the interface between the intrusion and the ambient fluid separates from a solid wall is considered. The fluid is assumed to be of finite depth and the incoming channel makes a downward angle $\alpha$ with the horizontal axis. This simple configuration provides a model for the plunging inflow and subsequent underflow of dense water in a reservoir or lake. Exact solutions are presented at infinite Froude number and compared with the solutions to the full nonlinear problem for supercritical flow. Limiting flows are found to separate from the upper boundary at a stagnation point, and regions of non-uniqueness in the solution domain are found.



## 1. Introduction

The behaviour of water flowing into reservoirs has attracted interest due to its importance in water quality management. Inflow of cold, polluted, salty or turbid water can significantly affect the behaviour of the water in the lake [1]. Extensive field and laboratory investigations of the flow in reservoirs included a study of the dynamics of riverine inflow (e.g., [2–4]). When river water enters a lake or a reservoir, the density of the water will usually be different from the density of the receiving lake surface water [5]. If the inflow is less dense than the surface water of the lake it will travel along the lake surface as an overflow [6]. However, if the inflowing water is denser than the surface water it will continue flowing downwards until it reaches the level of its neutral buoyancy [3], whereupon it leaves the river bed and intrudes horizontally into the ambient lake water forming an intrusion [5,7,8].

Forbes et al. [8,9] considered the flow of mid-level intrusions. They examined stratified intrusions with constant density in three layers in which both the top and the bottom layers are stationary and of finite depth, while the middle layer is in motion. Forbes et al. [8] analysed nonlinear periodic waves on the intrusion layer and found two branches of solution; at higher speeds both interfaces moved in phase, while at lower speeds they were out of phase. Solitary wave solutions were presented using both a full nonlinear method and KdV theory to produce weakly nonlinear solutions.

Hocking and Forbes [10] also considered this problem and explored the effect of the stream-bed slope when the water enters a reservoir and solutions were obtained for a range of inflow angles, Froude number (flow rate) and density differences. They showed that the steepness (amplitude divided by wavelength) of the waves on the interface depended on the angle of the bottom slope.

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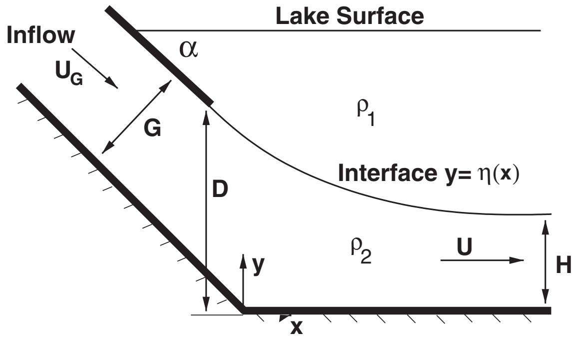


**Fig. 1.** Sketch showing the intrusion layer created when water enters a reservoir at angle $\alpha$, separating from the solid entry structure at height $D$ through a gap of size $G$. The downstream velocity is $U$ and depth approaches $H$.

Hocking [11] obtained a smooth separation solution for a flow when the entry angle is vertical ($\alpha = \pi/2$) and the moving layer is on the bottom of the channel. They found the interface separated smoothly until a limiting solution with a stagnation point formed. A number of authors have noted the existence of both smooth separation and stagnation point separation from solid boundaries [12–15].

The related problem of flow past a semi-infinite flat plate (i.e., $\alpha = 0$) has been studied recently by Vanden-Broeck [16], McCue and Forbes [17] and Ogilat et al. [18], who found subcritical ($F < 1$) wave solutions for flow past a semi-infinite flat plate in a fluid of finite depth. Maleewong and Grimshaw [19] considered the same problem with a plate depressed into the water using both nonlinear methods and KdV theory.

The present work is focused on intrusion flows along the bottom with a single interface as shown in Fig. 1. A simple model for inflow that assumes irrotational flow of inviscid fluid in two-dimensions is derived. The interface between the layers of different density is infinitesimally thin. The flow is from a solid structure submerged in the lake or reservoir and sloping downward with angle $\alpha$ where the interface separates from the upper surface.

In reality, the interface would be slightly diffuse, viscosity may play a role, real inflows are not two-dimensional and if the inflow has high speed then we would expect there to be some mixing at the interface. Benjamin [20] and Simpson [21] gave a thorough discussion of such effects. However, as a preliminary study the approximations made here are a good starting point. While our interest in this paper is on intrusion flows the problems are also relevant to weir and waterfall flows where the interface is between air and water. In those cases, the approximations involved are much less.

Results are obtained at supercritical flow rates down to the limiting minimal flow rate at which a stagnation point forms at the location of separation. All solutions found occur at supercritical (with Froude number greater than $F = 1$) flow rates and so no waves are found on the interface. Some non-uniqueness in the solution domain is identified. Solutions at sub-critical Froude numbers also exist, but these will be considered elsewhere.

## 2. Problem formulation

Consider the two-dimensional, steady, irrotational flow of an inviscid, incompressible fluid that intrudes horizontally beneath a layer of constant but lower density as shown in Fig. 1. In Cartesian coordinates, the elevation is $y$ and horizontal distance $x$, with the origin at the corner where the reservoir bottom meets the inflow structure. The receiving fluid is assumed to be homogeneous and the density of the fluid in this region is $\rho_1$ and in the intrusion itself $\rho_2$. The upper layer is assumed to be stagnant, but the bottom layer (of density $\rho_2$ with $\rho_2 > \rho_1$) moves so that in the downstream limit it approaches a speed $U$ and depth $H$. The fluid has an interface $y = \eta(x)$ above the intruding layer and the shape of it will be of particular interest. The intruding fluid is assumed to flow in from a structure consisting of two solid boundaries at a downward angle of $\alpha$, separated by a perpendicular distance $G$, and with separation of the free surface from the upper boundary at height $D$ as shown in Fig. 1. Thus, $G$ is the size of the gap between the walls of the inflow structure and if $U_G$ is the fluid speed high up in the structure, then $U_G G = UH$. Note that the point of separation need not be directly above the bottom corner point.

Since the fluid system is assumed to be incompressible and inviscid and the flow to be steady and irrotational, we can define a velocity potential $\phi(x, y)$ and streamfunction $\psi(x, y)$ that both satisfy Laplace's equation

$$\nabla^2\phi = 0 \quad \text{and} \quad \nabla^2\psi = 0, \tag{1}$$

within the flow domain, so that the velocity components $u$ and $v$ are given by

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \tag{2}$$

$$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{3}$$