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The basic attractor and the remainder of homo- and heteroclinic orbits☆

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ABSTRACT

In this paper, we discuss some issues in the dynamical systems theory of dissipative nonlinear partial differential equations (PDEs), on a bounded domain. A decomposition theorem says that attractors of PDEs can be decomposed into a basic attractor (a core) that attracts sets of positive measure, it attracts a prevalent set in phase space, and a remainder whose basin, up to sets that are attracted to the basic attractor, is shy, or of zero (infinite-dimensional) measure. If the basic attractor is low-dimensional and the remainder high-dimensional, then the dynamics can still be analyzed up to transients that are exponentially decaying toward the attractor in time. We focus on (ODE) examples of homo- and heteroclinic connections and show that generically these connections lie in the remainder but there exist exceptional cases where they lie in the basic attractor.

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1. Introduction

In this paper, we will give an an abbreviated introduction to the theory of *Basic Attractors and Control*, see [1]. We develop some central issues in the dynamical systems theory of dissipative nonlinear partial differential equations (PDEs), on a bounded domain, and connect it with the dynamical systems theory of ordinary differential equations (ODEs). The latter theory was developed during the latter half of the twentieth century and has revolutionized modern science and engineering.

The attempts to develop a dynamical systems theory for PDEs in the late twentieth century, see [2–6], had some success. It was established that dissipative nonlinear PDEs had finite-dimensional attractors. These are set of solutions that attract all other solutions in the phase space of the PDEs as time becomes large. But a troublesome gap remained between the dimension of the attractors obtained in estimates, see [4], and the dimensions of solutions observed in experiments and numerical simulations. This prevented a meaningful application of the theory. We explain how this problem was resolved and how the analysis of the core of the attractors, called *Basic Attractors*, can be reduced to the theory of ODEs and the bifurcation theory of their solutions, see [1].

John Milnor [7] had proven in 1985 that the attractors of ODEs have a decomposition if the Lebesgue measure of their basin of attraction is taken into account. This raised the question whether a similar decomposition could be found for the attractors of PDEs and whether the dimensions of their components that attracted positive measure in infinite-dimensional space were possibly small. Only these components were expected to play a role in most numerical simulations and experiments.

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The first success in this direction was proving, see [6], that the breather solutions of the damped and driven sine-Gordon equation attracted sets of positive measure by a cumbersome projection onto a finite-dimensional subspace of the phase space of the PDE. The breathers constitute the dynamically interesting part of the sine-Gordon basic attractor. The reason why this was relatively complicated is that there is no straight-forward analog of Lebesgue measure that can be defined on the whole of an infinite-dimensional space. However, Hunt et al. [8], developed a notion of measure zero, *shy*, and almost every, *prevalent*, in infinite dimensions. This set the stage for an elegant extension of Milnor's decomposition to attractors in infinite-dimensional space. After this development applications of the theory to many different types of PDEs became possible.

The decomposition theorem, Theorem 1 below, says that attractors of PDEs can be decomposed into a *basic attractor* (a core) that attracts sets of positive measure, indeed it attracts a prevalent set in phase space, and a *remainder* whose basin, up to sets that are attracted to the basic attractor, is shy. This offers an elegant closure of the gap discussed above. Namely, if the *basic attractor* is low-dimensional and the *remainder* high-dimensional, then the dynamics can still be analyzed up to transients that are exponentially decaying toward the attractor in time. The notion of the basin of attraction has to be generalized slightly and we do that by defining the *catchment* of a set below.

In [9,10] we illustrated the theory by an application to the viscous Moore–Greitzer equation, see Chapter 7 in [1], describing the air flow through a jet engine. The viscous Moore–Greitzer basic attractor turns out to contain the flow for the desired operation of the jet engine, called *design flow*, but also two undesirable instabilities in the flow called *surge* and *stall*. The complete qualitative description of the design flow and those two instabilities is a great accomplishment of the theory.

The theory of basic attractors is however a perfectly general theory and in [1] we lay the foundation for nonlinear heat, dissipative wave equations and dissipative nonlinear Schrödinger equations. These details will not be repeated in the paper but we refer the interested reader to Chapter 4 in [1].

In this paper, we define and prove the existence of the basic attractor, following Milnor's [7] results in finite dimensions. The decomposition of the global attractor, into a basic attractor and a remainder in infinite dimensions is proven. We refer the reader to [8] and [1] where the concepts of shy and prevalent sets, that are used in the definition of the basic attractor, are discussed in more detail. There we also give an example showing that there exist basic attractors of arbitrarily high dimensions. Finally, we construct a "typical" basic attractor that is low-dimensional, and a remainder that is high-dimensional, using the damped and driven sine-Gordon equation in Chapter 5 in [1].

In [11,12], the first impressive consequence of the theory of basic attractors is explored again by the example of the viscous Moore–Greitzer equation. The theory of basic attractors makes it possible to develop a *basic control theory* creating the means to control the surge and stall instabilities. Thus *basic attractors* lead to *basic control*, see [1].

In [1] and [13], Chapter 11, we also show how to approximate the general solution in the basic attractor by a finite basis of solutions from the basic attractor, these are called *basic approximations*. We prove in this chapter that the basic truncation basis consists of the first few Karhunen–Loeve (KL) [14] empirical eigenfunctions, when the motion is ergodic on the basic attractor. This was first observed numerically in [15]. Thus basic attractors provide an optimal low-dimensional truncation of the solutions to the nonlinear PDE and this also explains why the KL analysis works so surprisingly well on low-dimensional attractors of PDEs.

The numerical analysis of basic attractors has taken off in the last ten year with the availability of better numerical and symbolic programs, such as Matlab and Mathematica, see [1], especially in the engineering literature. Good examples are the papers on beam and shell dynamics by Awrejcewicz [16–23] and his collaborators.

2. The decomposition theorem

A global attractor $\mathcal{A}$ of an ODE or a PDE is defined to be the omega limit set of an absorbing set, see [1]. If the absorbing set is convex and the phase space of the dynamical system is $\mathbb{R}^n$ or a Banach space, the global attractor is connected and compact, see [1]. The problem with $\mathcal{A}$ is that it tends to be large and high-dimensional, see [1]. Thus it is desirable to find a more restrictive notion of an attractor that permits a decomposition of $\mathcal{A}$ into more manageable parts. The first step is to work with sets that are more general than basins, namely sets that can also be closed or neither open nor closed and look for the omega limits sets of these sets. In order to do this Milnor defined the realm of a set. Milnor gives the following definition on page 179 in [7]:

Definition 1. A closed subset $A \subset M$ (M a smooth compact manifold) will be called an attractor if it satisfies two conditions:

1. the realm of attraction $\rho(A)$ consisting of all points $x \in M$ for which $\omega(x) \subset A$, must have a strictly positive measure; and
2. there is no strictly smaller closed subset $A' \subset A$ such that $\rho(A')$ coincides with $\rho(A)$ up to a set of measure zero.

We want to generalize Milnor's notions to infinite dimensions and the first step is to define the generalization of the realm to infinite dimensions.

Definition 2. The catchment of a set A consists of all points x, such that $\omega(x) \subset A$.

Catchment is the generalization of a realm of a set. Thus a catchment can be infinite dimensional and in finite-dimensions the catchment becomes the realm. Next we have to generalize the notion of Lebesgue measure zero and "almost every" to infinite dimensions. The problem is that there is no analog of Lebesgue measure in infinite dimensions. It is easy to see,

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