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Multilayer diffusion in a composite medium with imperfect contact

Natalie E. Sheils

Institute for Math and Its Applications, 207 Church Street SE, 306 Lind Hall, Minneapolis, MN 55455, United States

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ABSTRACT

The problem of heat conduction in one-dimensional piecewise homogeneous composite materials is examined by providing an explicit solution of the one-dimensional heat equation in each domain. The location of the interfaces is known, but neither temperature nor heat flux are prescribed there. We find a solution using the Unified Transform Method, due to Fokas and collaborators, applied to interface problems and compute solutions numerically.

1. Introduction

The problem of heat conduction in a composite wall is a classical problem in design and construction. It is usual to restrict to the case of walls with physical properties that are constant throughout the material and are considered to be of infinite extent in the directions parallel to the wall. Further, we assume that temperature and heat flux do not vary in these directions. In that case, the mathematical model for heat conduction in each wall layer is given by Hahn and Özisik [1, Chapter 10]:

$$u_t^{(j)} = \kappa_j u_{xx}^{(j)}, \quad x_{j-1} < x < x_j, \tag{1a}$$

$$u^{(j)}(x,0) = u_0^{(j)}(x), \quad x_{j-1} < x < x_j, \tag{1b}$$

$$\beta_1 u^{(1)}(x_0,t) + \beta_2 u_x^{(1)}(x_0,t) = f_1(t), \quad t > 0, \tag{1c}$$

$$\beta_3 u^{(n+1)}(x_{n+1},t) + \beta_4 u_x^{(n+1)}(x_{n+1},t) = f_2(t), \quad t > 0, \tag{1d}$$

where $u^{(j)}(x, t)$ denotes the temperature in the wall layer indexed by (j), $\kappa_j > 0$ is the heat-conduction coefficient of the jth layer (the inverse of its thermal diffusivity), $x = x_{j-1}$ is the left extent of the layer, $x = x_j$ is its right extent, and β_n for $n = 1, 2, 3, 4$ are constants. The sub-indices denote derivatives with respect to the one-dimensional spatial variable x and the temporal variable t. The function $u_0^{(j)}(x)$ is the prescribed initial condition of the system. The continuity of the temperature $u^{(j)}$ and of its associated heat flux $\kappa_j u_x^{(j)}$ are imposed across the interface between layers. In what follows it is convenient to use the quantity σ_j, defined as the positive square root of κ_j: $\sigma_j = \sqrt{\kappa_j}$.

E-mail address: nesheils@umn.edu

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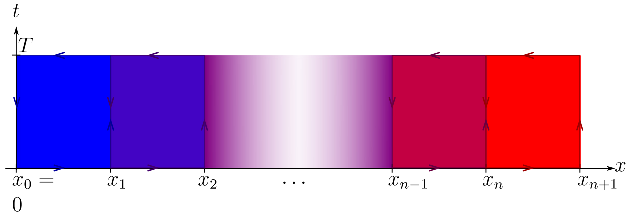

Fig. 1. Domains for the application of Green's Theorem in the case of a finite domain with n interfaces.

If each layer is in perfect thermal contact then the interface conditions are

$$u^{(j)}(x_j,t) = u^{(j+1)}(x_j,t), \quad t>0, \tag{2a}$$

$$\sigma_j^2 u_x^{(j)}(x_j,t) = \sigma_{j+1}^2 u_x^{(j+1)}(x_j,t), \quad t>0. \tag{2b}$$

A derivation of the interface conditions for perfect thermal contact is found in [1, Chapter 1]. However, if the thermal contact is imperfect we prescribe the interface conditions

$$\sigma_j^2 u_x^{(j)}(x_j,t) = H_j\big(u^{(j+1)}(x_j,t) - u^{(j)}(x_j,t)\big), \quad t>0, \tag{3a}$$

$$\sigma_{j+1}^2 u_x^{(j+1)}(x_j,t) = H_j\big(u^{(j+1)}(x_j,t) - u^{(j)}(x_j,t)\big), \quad t>0, \tag{3b}$$

where $H_j \neq 0$ is the contact transfer coefficient at $x=x_j$ and $1 \le j \le n$. Perfect thermal contact, is recovered in the limit $H_j \to \infty$. In applications, imperfect boundary conditions are used to model roughness and contact resistance [2–5]. Carr and Turner [3] approach this problem using a semi-analytical method based on the Laplace transform and an orthogonal eigenfunction expansion. Their interest in the problem is to accurately solve a two-scale modeling problem for transport or fluid flow in porous media exhibiting small scale heterogeneities in material properties. The authors note that for a large number of layers, multilayer diffusion is possibly the most simple example of such a problem. However, their numerical implementation for their analytical solution only works for up to ten layers [3]. They also propose a "semi-analytical" model which works for a large number of layers.

In this paper, we use the Fokas Method (also called the Unified Transform Method) [6–8] to provide explicit solution formulae for different heat transport interface problems of the types described above. Even for a simple problem (two finite walls in perfect thermal contact), the classical approach using separation of variables [1] can provide an explicit answer only implicitly. Indeed, the solution obtained in [1] depends on certain eigenvalues defined through a transcendental equation that can be solved only numerically. In contrast, the Fokas Method produces an explicit solution formula involving only known quantities. In [9] the problem of heat conduction in perfect thermal contact was considered using the Fokas Method to provide explicit solution formulae for a number of examples for up to three domains. In this paper we extend that method to include more general interface conditions and a generic number of interfaces.

Interface problems for partial differential equations (PDEs) are initial boundary value problems for which the solution of an equation in one domain prescribes boundary conditions for the equations in adjacent domains. In applications, interface conditions are often obtained from conservation laws [10]. Few interface problems allow for an explicit closed-form solution using classical solution methods. Using the Fokas Method, such solutions may be constructed for both dissipative and dispersive linear interface problems as shown in [9,11–16].

2. The Fokas Method for the heat equation

We follow the standard steps in the Fokas Method. Assuming existence of a solution, we begin with the so-called "local relations":

$$\big(e^{-ikx+\omega_j(k)t}u^{(j)}\big)_t = \big(e^{-ikx+\omega_j(k)t}\sigma_j^2(u_x^{(j)} + iku^{(j)})\big)_x, \tag{4}$$

where $\omega_j(k) = (\sigma_j k)^2$. Without loss of generality we shift the problem so that $x_0 = 0$.

Integrating each local relation (4) around the appropriate domain (see Fig. 1) and applying Green's Theorem we find the global relations:

$$\begin{aligned} 0 = {} & \int_{x_{j-1}}^{x_j} e^{-ikx}u_0^{(j)}(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} e^{-ikx+\omega_j(k)T}u^{(j)}(x,T)\,\mathrm{d}x \\ & + \int_0^T \sigma_j^2 e^{-ikx_j+\omega_j(k)s}(u_x^{(j)}(x_j,s) + iku^{(j)}(x_j,s))\,\mathrm{d}s \\ & - \int_0^T \sigma_j^2 e^{-ikx_{j-1}+\omega_j(k)s}(u_x^{(j)}(x_{j-1},s) + iku^{(j)}(x_{j-1},s))\,\mathrm{d}s, \end{aligned} \tag{5}$$