Applied Mathematical Modelling 45 (2017) 238–254

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

# A new semi-analytical collocation method for solving multi-term fractional partial differential equations with time variable coefficients

S.Yu. Reutskiy[a,b,*]

[a] College of Mechanics and Materials, Hohai University, Nanjing 210098, PR China
[b] Institute of Technical Problems of Magnetism of the National Academy of Sciences of Ukraine, 61106, Kharkov, Ukraine

ARTICLE INFO

*Article history:*
Received 10 June 2016
Revised 28 October 2016
Accepted 21 December 2016
Available online 27 December 2016

*Keywords:*
Time fractional equations
Numerical solution
Wave-diffusion equation
Time-fractional telegraph equation
Fractional sub-diffusion equations

ABSTRACT

The aim of this paper is to present a new numerical method for solving a wide class of fractional partial differential equations (FPDEs) such as wave-diffusion equations, modified anomalous fractional sub-diffusion equations, time-fractional telegraph equations. The proposed method is based on the Fourier series expansion along the spatial coordinate which transforms the original equation into a sequence of multi-term fractional ordinary differential equations (ODEs). These fractional equations are solved by the use of a new efficient numerical technique – the backward substitution method. The numerical examples confirm the high accuracy and efficiency of the proposed numerical scheme in solving FPDEs with variable in time coefficients.



## 1. Introduction

In this paper we present a novel method for the numerical solution of the time fractional order equation which can be written in the form:

$$D_t^{(\nu)} u(x,t) + \sum_{i=1}^{I} \alpha_i(t) D_t^{(\nu_i)} u(x,t) \tag{1}$$

$$= \left[ \sum_{i=I+1}^{K} \alpha_i(t) D_t^{(\nu_i)} \right] \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \;\; 0 \le x \le a, \;\; 0 \le t \le T. \tag{1}$$

We consider this equation with the boundary conditions along the spatial coordinate:

$$u(0,t) = g_0(t), \;\; u(a,t) = g_1(t), \tag{2}$$

and initial conditions:

$$u(x,0) = h_0(x), \;\; \frac{\partial u}{\partial t}(x,0) = h_1(x). \tag{3}$$

* Correspondence to: College of Mechanics and Materials, Hohai University, Nanjing 210098, PR China.
*E-mail addresses:* sergiy.reutskiy2016@yandex.ru, sergiyreutskiy@gmail.com

http://dx.doi.org/10.1016/j.apm.2016.12.029
0307-904X/

Here $0 < \nu \le 2$, $0 \le \nu_i < \nu$, $i = 1, \dots, K$ – are fractional or integer numbers; $\alpha_i(t)$, $g_0(t)$, $g_1(t)$, $h_0(x)$, $h_1(x)$ are known smooth enough functions. The second condition in (2) and (3) is applied only when $\nu > 1$. In case $\nu > 1$ we also consider the boundary conditions along the time coordinate $t$ on the right hand side of the interval [0, $T$]:

$$u(x,0) = h_0(x), \;\; u(x,T) = h_T(x). \tag{4}$$

Throughout the paper we consider the fractional Caputo derivatives which are defined as follows [1–3]:

$$D^{(\nu)} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\nu)} \displaystyle\int_0^x \dfrac{f^{(n)}(t)}{(x-t)^{\nu-n+1}}, & n-1 < \nu < n, \\ f^{(n)}(x), & \nu = n, \end{cases} \tag{5}$$

where $n \in \mathcal{N} = \{1, 2, \dots\}$ is the set of positive integers, and $\Gamma(z)$ denotes the gamma function. In particular, for the power functions we get:

$$D^{(\nu)} x^p = \begin{cases} 0, & \text{if } p \in \mathcal{N}_0 \text{ and } p < n, \\ \dfrac{\Gamma(p+1)}{\Gamma(p+1-\nu)} x^{p-\nu}, & \text{if } p \in \mathcal{N}_0 \text{ and } p \ge n \text{ or } p \notin \mathcal{N}_0 \text{ and } p > n-1, \end{cases} \tag{6}$$

where $\mathcal{N}_0 = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. This formula is widely used throughout the paper.

Let us note that Eq. (1) includes many different known equations as particular cases. For example:

- the time-fractional sub-diffusion equation [4]:

$$D_t^{(\nu)} u(x,t) = a \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \;\; 0 < \nu < 1, \;\; a > 0, \tag{7}$$

- the time-fractional telegraph equation [5–7]:

$$D_t^{(\nu)} u(x,t) + \alpha_1 D_t^{(\nu-1)} u(x,t) + \alpha_2 u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{8}$$

- the multi-term time-fractional diffusion and diffusion-wave equations [8–10]:

$$D_t^{(\nu)} u(x,t) + \sum_{i=1}^{n} \alpha_i D_t^{(\nu_i)} u(x,t) = K_e \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \;\; 0 < \nu_i < \nu < 2, \;\; \alpha_i, K_e > 0, \tag{9}$$

- the time-fractional modified anomalous sub-diffusion equation [11,14,15]:

$$\frac{\partial u(x,t)}{\partial t} = \left[\alpha_1 D_t^{(1-\nu_1)} + \alpha_2 D_t^{(1-\nu_2)}\right] \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \;\; 0 < \nu_1, \nu_2 < 1, \;\; \alpha_1, \alpha_2 > 0. \tag{10}$$

Fractional differential equations arise in various areas of science and engineering. The advantages of FDEs become apparent in modeling electrical properties of real materials, as well as in description of blood flow, and in many other fields (see, for example, [2,16,17] and references cited therein). Partial differential equations of fractional order play an important role in modeling the so-called anomalous transport phenomena and in the theory of complex systems [16,18–20]. The time-fractional diffusion-wave equation describes important physical phenomena that arise in amorphous, colloid, glassy and porous materials, in fractals and percolation clusters, dielectrics and semiconductors, biological systems, polymers, random and disordered media, geophysical and geological processes [13,18,20–23]. In [24] the fractional-order telegraph equation was used to model neutron transport in a nuclear reactor with slab geometry.

Due to the growing applications, a considerable attention has been given to development of the numerical techniques for FPDEs. They could be divided into the following groups.

The first group includes the methods that use discretization of any kind (finite difference method, finite elements method etc.). An implicit conditionally stable difference scheme for solution modified anomalous sub-diffusion equations was proposed by Liu et al. [11]. By using the energy method, they showed that the convergence order of the method is $O(\tau + h^2)$. Here $\tau$ and $h$ denote the space and time step sizes, respectively. A high-order compact finite difference scheme for solving one-dimensional fractional diffusion equations was proposed by Cui [25] with the stability and convergence analysis. Liu et al. [12] proposed a semi-discrete approximation and a full discrete finite element approximation for solving modified anomalous sub-diffusion equations in a finite domain. They proved the stability and convergence of the proposed methods. An unconditionally stable difference scheme of order $O(\tau + h^4)$ for solving the same equations was proposed by Mohebbi et al. [14]. Liu et al. [8] developed several finite difference schemes for solving the multi-term time fractional advection–dispersion and wave-diffusion equations. These schemes are based on the fractional predictor–corrector method. Hu and Zhang [26] proposed a finite difference scheme for an approximate solution of the fourth-order fractional diffusion-wave equations. Two fully discrete schemes were proposed by Zeng et al. [4] for time-fractional sub-diffusion equations. This technique uses a space discretization by the finite element method and a time discretization by fractional linear multistep methods. Compact finite difference schemes were studied by Wang and Vong [27] in application to modified anomalous fractional sub-diffusion equations and fractional diffusion-wave equations. The methods for the numerical solution of multi-term time fractional partial differential equations proposed by Dehghan et al. [10] also falls into this group. The time fractional derivatives are approximated by a scheme of the order $O(\tau^{3-\alpha})$, $1 < \alpha < 2$ and the space derivative is discretized