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A full-scale investigation of the directional returns to scale in data envelopment analysis

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A R T I C L E I N F O

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Returns to scale is considered as one of the important concepts in data envelopment analysis (DEA) which can be useful for deciding to increase or decrease the size of a particular decision making unit. Traditional returns to scale on the efficient surface of the production possibility set with variable returns to scale (VRS) technology is introduced as a ratio of proportional changes of output components to proportional changes of input components. However, a problem which may arise in the real world is the impossibility or undesirability of proportional change in the input or output components. One of the attempts which is made to solve the aforementioned problem is the work of Yang et al., 2014. They have introduced the "directional returns to scale" in the DEA framework and have proposed some procedures to estimate and measure it. In this paper, the introduced directional returns to scale is investigated from a new perspective based on the defining hyperplanes of the production possibility set with VRS technology. We propose some algebraic equations and linear programming models which in addition to measuring the directional returns to scale, they enable us to analyse it. Moreover, we introduce the concepts of the best input and output direction vectors for expansion of input components or compression of output components, respectively, and propose two linear programming models in order to obtain these directions. The presented equations and models are demonstrated using a case study and numerical examples.

1. Introduction

Data envelopment analysis (DEA) is a technique for measuring the efficiency of a set of homogeneous decision making units (DMUs) which was presented for the first time by Charnes et al. [7]. Since then, extensive research in data envelopment analysis was performed and the scope of this branch of science has been broadened and more concepts have been introduced. Returns to scale (RTS) is one of the important concepts in DEA which can be useful for a manager when he (she) wants to decide to increase or decrease the size of a particular decision making unit; in other words, "RTS can provide useful information on the optimal size of DMUs" [10]. In economics, sometimes RTS was used as elasticity concept [19,23]. In a technology with one input and one output, scale elasticity is defined as the ratio of the marginal productivity to the average productivity at an efficient unit. In the neoclassical economics, the concept of scale elasticity was developed into technologies with multiple inputs and outputs (see [14] and [23]).

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RTS in DEA was discussed for the first time by Banker [2] and also Banker et al. [3] who provided methods to estimate RTS. Since then, there have been remarkable efforts in defining the RTS within the DEA context (e.g. [4,5,8,9,17,24–26]).

It is obvious that RTS gives useful information to managers to change the size of a particular efficient DMU but there are two major problems in the traditional RTS. The first one is that either the change of input or output components may not be desirable proportionally or there is no possibility for the proportional change of components in practice. According to this defect, the traditional RTS cannot be utilized in most of the real practical cases. The second problem is the local property of the RTS, i.e., RTS measure exhibits the rate of proportional change in output components with respect to the proportional change of input components in the local sense[1]. Because of this property, the obtained information about the RTS may be insufficient for the manager to make a decision about resizing the DMU operation.

Regarding the first problem, Fukuyama [11] presented a directional technology scale elasticity formula, which accommodates Farrell input and output scale elasticities as special cases. Podinovski and Forsund [20] used theorems related to the derivative of the optimal value function, extended the traditional RTS and introduced a new one. They assumed that input and output components can be divided into three disjoint sets. Then, they investigated the response of the factors included in the second set to marginal changes of the factors in first set under the assumption that the factors in third set remain constant. In another attempt in this area, Balk et al. [1] viewed the scale elasticity as a directional derivative measure and then for any functional representation of the technology and at a given point in the input-output space, they computed scale elasticity along any direction.

In addition to aforementioned studies, Yang [32] introduced the concept of directional returns to scale (DRTS), and afterwards, Yang et al. [34] investigated and illustrated it by studying biological institutes of the Chinese Academy of Sciences (CAS). Further, Yang et al. [33] proposed a definition of DRTS in the DEA framework. They provided a procedure for estimating DRTS of strongly efficient DMUs based on the Finite Difference Method (FDM) proposed by Rosen et al. [21] and also Golany and Yu [12]. They also proposed a linear programming model to calculate the upper and lower bounds of the directional scale elasticity (SE). In fact, the introduced DRTS calculates the ratio of output components' changes along a particular output direction vector to fit input components' changes along a particular input direction vector on the efficient production surface.

It is well known that T_V is a polyhedron and it is sometimes needed to obtain its defining hyperplans. There are many approaches to find the weak and strong defining hyperplanes of T_V. For instance, the works of Jahanshahloo et al. [16] and Jahanshahloo et al. [15] can be mentioned as they obtained specific type of defining hyperplans. The current paper, investigates the DRTS introduced by Yang et al. [33] in a new perspective based on the defining hyperplanes of VRS production possibility set (T_V). In fact, some algebraic equations are proposed which in addition to measuring the DRTS, they enable us to analyse the DRTS. Using the presented equations, we can also seek the precise relation between the changes of output components and changes of input components along the specific input and output directions. In this way, the problem of local property in DRTS can be solved, too. However, it is known that obtaining the defining hyperplanes of T_V is a highly computational process. Therefore, to solve this problem, the presented equations are converted to linear programming models which enable us to measure and analyse the DRTS without using the defining hyperplanes. Moreover, we introduce the concepts of best input direction vector for expansion of the input components and best output direction vector for compression of the output components and propose two new linear programming models in order to obtain these direction vectors.

Here, it is worth to mention that all of the presented concepts and models can just deal with decision making units which are considered as whole units (or black box). Thus, for example, using them to deal with the bi-level decision systems (DMUs), which have been discussed by Wu et al. [29] or Wu [28], does not seem to be perfect. The reason is that the DMUs include two decentralized sub-systems with a particular hierarchical structure, while in applying the presented concepts or models, they should be considered as whole units without any inner intermediate inputs/outputs and any decision hierarchy. In such cases, developing a new concept from the returns to scale (and then directional returns to scale) based on the DMUs' special structure seems better than using the presented concepts or models (e.g. see [36]). Thus, in this way, all of the internal information about the DMUs can be used to achieve better results.

The rest of this paper is organized as follows: in Section 2, the definitions of traditional and directional returns to scale are presented. Moreover, in this section, the pointwise minimum function (of linear functions) is briefly discussed and two important theorems of it are described which are bases of the current paper. Section 3, with the assumption that the defining hyperplanes of T_V are available, propose a number of equations to measure and conduct a complete analysis of the DRTS. In Section 4, the equations proposed in Section 3 are converted to linear programming models. These models do not require the existence of defining hyperplanes of T_V. Furthermore, in this section, the concepts of best input and output direction vectors are introduced for expansion of input components or compression of output components, respectively, and two linear programming models are proposed in order to obtain them. Section 5 contains two numerical examples to illustrate the proposed theories and models. Moreover, their practical relevance in the real world is shown using a small case study in this section. The conclusion is summarized in the last section.

[1] Hadjicostas and Soteriou [13] present a theoretical framework for one-sided RTS