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Determination of Robin coefficient in a fractional diffusion problem

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ABSTRACT

This paper investigates a nonlinear inverse problem associated with a fractional diffusion equation for identifying a Robin coefficient in the boundary conditions from a boundary measurement. The existence and uniqueness of a weak solution for the corresponding direct problem is provided. We formulate the inverse problem into a regularized variational problem and deduce the gradient of the regularization functional based on an adjoint problem. Then the standard conjugate gradient method is employed to solve the variational problem. The numerical results for three examples are presented to illustrate the efficiency of the proposed algorithm.

1. Introduction

Fractional calculus have attracted wide attentions in recent years and have been successfully applied to problems in system biology [1], physics [2–4], chemistry and biochemistry [5], hydrology [6], medicine [7,8] and finance [9,10].

Time fractional diffusion equations are deduced by replacing the standard time derivative with a time fractional derivative and can be used to describe superdiffusion and subdiffusion phenomena [11–13]. The direct problems for time-fractional diffusion equations have been studied extensively in recent years [14–17]. However, the inverse problems for time fractional diffusion equations are still in the initial steps and not too many papers can be found [17–22]. Recently some nonlinear fractional inverse problems have been investigated in [23–27].

In this paper, we consider the following inverse Robin coefficient problem for a fractional diffusion equation:

$$\begin{cases} {}_0\partial_t^\alpha u(x,t) - u_{xx}(x,t) = 0, & x\in(0,1), \quad t\in(0,T], \quad 0<\alpha<1,\\ (-u_x(x,t)+\sigma(t)u(x,t))|_{x=0} = g_1(t), & t\in(0,T],\\ (u_x(x,t)+\sigma(t)u(x,t))|_{x=1} = g_2(t), & t\in(0,T],\\ u(x,0)=\varphi(x), & x\in[0,1], \end{cases} \tag{1.1}$$

where ${}_0\partial_t^\alpha$ is the Caputo fractional derivative of order α defined by:

$${}_0\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}d\tau, \quad 0<\alpha<1, \tag{1.2}$$

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and $g_1(t)$, $g_2(t) \in L^2(0, T)$, $\varphi(x) \in H^2(0, 1)$ are given functions, $\sigma \in \mathcal{S} = \{\sigma \in L^\infty(0,T) | \overline{\sigma} \geq \sigma \geq \underline{\sigma} > 0, a.e.\ t \in (0,T)\}$ is a Robin coefficient describing the convection between the solute in a body and one in the ambient environment,where $\overline{\sigma}$ and $\underline{\sigma}$ are two known positive constants. If $\sigma(t)$ is given, the problem (1.1) is a well-posed direct problem, see Section 3 for the existence and uniqueness of a weak solution. As for the high dimensional case, the existence and uniqueness of classical solution has been studied in [28].

Here, the inverse Robin coefficient problem is to find $\sigma(t)$ from an additional boundary condition

$$u(0,t) = h(t).$$

Under a special assumption for σ, we can know the measurement data $h(t)$ is enough for determining $\sigma(t)$ in (1.1) in the sense of classical solution, see [29] and for a general coefficient σ, we have no idea to prove the uniqueness for the inverse Robin coefficient problem. This is an open problem.

For the integer-order case, this inverse Robin coefficient problem has been widely studied. Yang et al. [30] and Jin et al. [31] used conjugate gradient methods to solve this problem. Yan et al. [32] used a Bayesian inference approach to identify a Robin coefficient in one-dimensional parabolic problems. In [33], Lenhart and Wilson presented an optimal control formulation of the inverse problem, established the well-posedness of the formulation, and derived the optimal system. Slodicka and Van Keer [34] studied the recovery of a Robin coefficient in a semilinear parabolic equation from an over-specified non-local boundary condition, and proposed a time discretization by Rothes method with some convergence analysis. Slodicka et al. [35] extended the analysis to estimating a temporally-dependent Robin coefficient in a nonlinear boundary condition for one-dimensional heat equation, and showed the existence and uniqueness of the solution.

To our knowledge, there are no literatures to give any theoretical and numerical results for fractional diffusion equations. The difficulty may come from the nonlinearity and long memory properties of the fractional derivative.

In this paper, we focus mainly on a numerical method for recovering the Robin coefficient in the fractional diffusion problem (1.1). We transform the inverse problem into a regularized variational problem and deduce the gradient of the regularization functional based on an adjoint problem. Then the standard conjugate gradient method is employed to solve the variational problem and the Moronov discrepancy principle is applied to find a suitable stopping step. We use a finite element method to solve the direct problems in each iteration.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries used in Section 4. The existence and uniqueness of a weak solution for the direct problem is given in Section 3. In Section 4, we formulate the inverse problem into a variational problem and deduce the gradient of the regularized functional. Section 5 is devoted to present a conjugate gradients algorithm. The numerical implementation for the direct problem is given in Section 6. Numerical results for three examples are presented and discussed in Section 7. Finally, concluding remarks are given in Section 8.

2. Preliminary

Let $AC[0, T]$ be the space of functions f which are absolutely continuous on $[0, T]$. Throughout this paper, we use the following definitions and propositions, see [36,37].

Definition 2.1. Let $z(t) \in AC[0, T]$. The Caputo fractional left derivative of order α is defined by:

$$_0\partial_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z'(s)}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 < t \leq T. \tag{2.3}$$

The Caputo fractional right derivative of order α is defined by:

$$_t\partial_T^\alpha z(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{z'(s)}{(s-t)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 \leq t < T. \tag{2.4}$$

Definition 2.2. If $z(t) \in L(0, T)$, then the Riemann–Liouville fractional left integral of order α is defined by:

$$_0I_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{z(s)}{(t-s)^{1-\alpha}} ds, \quad 0 < \alpha < 1, \quad 0 < t \leq T. \tag{2.5}$$

Definition 2.3. Let $z(t) \in AC[0, T]$. The Riemann–Liouville fractional left derivative of order α is defined by:

$$_0D_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{z(s)}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 < t \leq T. \tag{2.6}$$

The Riemann–Liouville fractional right derivative of order α is defined by:

$$_tD_T^\alpha z(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{z(s)}{(s-t)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 \leq t < T. \tag{2.7}$$

Proposition 2.1. *Let $z(t) \in AC[0, T]$. Then the Caputo fractional left derivative $_0\partial_t^\alpha z(t)$ and the Riemann–Liouville left fractional derivative $_0D_t^\alpha z(t)$ exist almost everywhere on $[0, T]$, there is a relationship between the Caputo fractional left derivative and the Riemann–Liouville left fractional derivative.*

$$_0D_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \frac{z(0)}{t^\alpha} + {_0\partial_t^\alpha z(t)}, \quad 0 < \alpha < 1.$$