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An enhanced tensorial formulation for elastic degradation in micropolar continua

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ABSTRACT

In the past, a lot of applications of the micropolar (or Cosserat) continuum theory have been proposed, especially in the field of granular materials analysis and for strain localization problems in elasto-plasticity, due to its regularization properties. In order to make possible the application of the micropolar theory to different constitutive models and to extend its regularization properties also to damage models, in this work a general formulation for elastic degradation based on the micropolar theory is proposed. Such formulation is presented in a unified format, able to enclose different kinds of elasto-plastic, elastic-degrading and damage constitutive models. A peculiar tensor-based representation is introduced, in order to guarantee the conformity with analogous theories based on the classic continuum, in such a way as to make possible the application to the micropolar theory of theoretical and numerical resources already defined for the classic theory. Peculiar micropolar scalar damage models are also proposed, and derived within the new general formulation.

1. Introduction

In the past, a lot of efforts have been made in order to define a unified formulation for constitutive models based on the classic continuum theory. Such a formulation has been introduced as a theoretical resource able to describe different elasto-plastic, elastic-degrading and damage constitutive models in a common framework [1–4].

The unified theory is based on a proper tensorial formalism that is not only considered as a means for the representation of the involved equations, but also as a fundamental resource for the development of the constitutive models [5].

As pointed out by a number of authors, the standardization offered by the use of a unified theoretical framework provides a common language, that allows an easy comparison between different constitutive models and a straightforward combination of damage and plastic effects, for example. Furthermore, such a unified framework allows to extend to elastic-degradation well known theoretical resources already developed in the field of multi-surface plasticity, like closed-form solutions of strain localization analysis, for example [6–8].

Recently, the computational advantages of the unified formulation have been investigated by Penna [9]. The inclusion of the tensor-based representation of the unified theory in a numerical code allows to optimize the implementation of elasto-plastic and elastic-degrading constitutive models with different levels of anisotropy, since the different models may

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share common theoretical and computational resources. Furthermore, it has been shown that the computational framework proposed by the author [9] makes the constitutive models implementation independent on both the analysis model (e.g., three-dimensional, plane-stress, plane-strain, etc.) and on the adopted numerical method (e.g., finite element method, generalized finite element method, boundary element method, and mesh-free methods) [10–12].

Despite the wide number of applications of the micropolar theory to elasto-plasticity (see, e.g., [13–16]), only a few works are devoted to its combination with damage models [17–19]. In both the approaches to micropolar media constitutive description, the proposed models are not derived within a more general unified formulation, like the one of the classic continuum theory.

A first attempt of inclusion of the micropolar theory into a unified framework for elasto-plasticity can be found in Gori et al. [20], while a basic approach to damage has been investigated in [21,22], where scalar damage models for micropolar media have been proposed. The aforementioned papers also point out the problem of consistency between the unified formulation for the classic continuum theory and the one of the micropolar theory; such a consistency represents a central issue in the definition of a proper unified formulation for micropolar media, since it allows a straightforward extension to the micropolar theory of theoretical and computational resources already defined for the classic theory.

This paper presents, to the authors knowledge, one of the first attempts of extension to the micropolar theory of a general elastic-degrading unified formulation. Micropolar media are here presented in a unified framework able to encompass, like its analogous for the classic theory, a large amount of elasto-plastic and damage models. Peculiar attention is devoted to the consistency with the unified formulation of the classic theory. Such an objective is achieved by means of a proper *enhanced tensorial formulation* for the representation of the micropolar theory. The first part of the paper recalls the basic equations regarding micropolar media. In the second part, a formulation for elastic-degradation in micropolar continua is proposed. Such a formulation is then reformulated making use of the aforementioned *enhanced tensorial formulation*, in order to guarantee a formal consistency with the classic theory, both from a theoretical and computational point of view. Peculiar scalar damage models are then derived within the proposed general framework. Finally, numerical simulations performed in the computational system **INSANE** (INteractive Structural ANalysis Environment) [23] are presented in order to illustrate the proposed models.

1.1. Notations

Some standard notations used in the body of the paper are summarized here. The symbol $\mathbf{D} \subseteq \mathbf{E}$ indicates the domain of the body, i.e., a subset of the three-dimensional Euclidean space $\mathbf{E}$, in which the orthonormal basis $(\bar{e}_i)$ is defined. *Vectors* are indicated as $\bar{x} = x_i\ \bar{e}_i$, while *second-order* and *fourth-order* tensors respectively as $\underline{x} = x_{ij}\ \bar{e}_i \otimes \bar{e}_j$, and $\hat{\mathbf{x}} = x_{ijk\ell}\ \bar{e}_i \otimes \bar{e}_j \otimes \bar{e}_k \otimes \bar{e}_\ell$. The symbol $\cdot$ denotes both the standard dot product between vectors and the total contraction between tensors like, for example, $\bar{x} \cdot \bar{y} = x_i\ y_i$, $\underline{x} \cdot \bar{y} = x_{ij}\ y_j\ \bar{e}_i$, $\hat{\mathbf{x}} \cdot \underline{y} = x_{ijkl}\ y_{kl}\ \bar{e}_i \otimes \bar{e}_j$ and the other possible combinations. With the symbol $\otimes$, the standard tensorial product, as $\bar{x} \otimes \bar{y} = x_i\ y_j\ \bar{e}_i \otimes \bar{e}_j$ or $\underline{x} \otimes \underline{y} = x_{ij}\ y_{k\ell}\ \bar{e}_i \otimes \bar{e}_j \otimes \bar{e}_k \otimes \bar{e}_\ell$, is indicated. In the following of the paper, if not differently specified, spaces will be assumed to be three-dimensional and the latin indexes will run from 1 to 3. In the case of *generalized* quantities, defined in six-dimensional spaces, greek letters will be used to indicate indexes running from 1 to 6. In some applications, in order to simplify the treatise, the Voigt notation will be used to represent second-order and fourth-order tensors; once a certain coordinates system has been fixed, a generic second-order tensor $\underline{x}$ with dimension three can be represented by means of an *array* with nine components, indicated with the symbol $\{\ \underline{x}\ \}$. In an analogous way, a fourth-order tensor $\hat{\mathbf{x}}$ with dimension three can be represented by means of a 9×9 matrix, indicated as $[\hat{\mathbf{x}}]$.

2. Micropolar media

In a geometrically-linear context, the configuration of a micropolar continuum is characterized, at each point of the domain $\mathbf{D}$, by a *displacement field* $\bar{u}$ and a *micro-rotation field* $\bar{\varphi}$, leading to the following strain measures:

$$\underline{\gamma} = \mathrm{grad}^{\mathrm{T}}(\bar{u}) - \mathsf{e} \cdot \bar{\varphi} = \left(u_{j,i} - \mathsf{e}_{ijk}\ \varphi_k\right)\ \bar{e}_i \otimes \bar{e}_j, \tag{1}$$

$$\underline{\kappa} = \mathrm{grad}^{\mathrm{T}}(\bar{\varphi}) = \varphi_{j,i}\ \bar{e}_i \otimes \bar{e}_j, \tag{2}$$

which are referred to, respectively, as *strain tensor* and *micro-curvature tensor*, and where the symbol e indicates the standard *Levi-Civita* operator with three indexes. To these strain measures correspond, respectively, the stress tensor $\underline{\sigma}$ and the couple-stress tensor $\underline{\mu}$ which must satisfy, in a *quasi-static* context, the local equilibrium equations for forces and moments in the domain $\mathbf{D}$

$$\mathrm{div}^{\mathrm{T}}(\underline{\sigma}) + \bar{b} = \bar{0} \longrightarrow \left(\sigma_{ij,i} + b_j\right)\bar{e}_j = \bar{0}, \tag{3}$$

$$\mathrm{div}^{\mathrm{T}}(\underline{\mu}) + \mathsf{e} \cdot \underline{\sigma} + \bar{l} = \bar{0} \longrightarrow \left(\mu_{ij,i} + \mathsf{e}_{jk\ell}\ \sigma_{k\ell} + l_j\right)\ \bar{e}_j = \bar{0}, \tag{4}$$

where $\bar{b}$ and $\bar{l}$ represent, respectively, volume forces and volume couples acting in the body domain. To the previous equations, the following *natural* and *essential* boundary conditions are associated

$$\bar{n} \cdot \underline{\sigma} = \bar{t}_A \text{ at } \partial\mathbf{D}_n^u, \quad \bar{n} \cdot \underline{\mu} = \bar{t}_C \text{ at } \partial\mathbf{D}_n^\varphi, \tag{5}$$