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Electrodynamic-mechanical boundary value problems and gauge transformations in rigid dielectrics with constitutive magnetoelectric coupling

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ABSTRACT

Gauge transformations are generally applied to decouple Maxwell's equations, introducing gauge invariant vector and scalar potentials in connection with suitable gauge differential equations. This step simplifies the analytical or numerical solution of electrodynamic boundary value problems. In dielectrics, the propagation of electromagnetic waves is usually investigated restricting the problems to simple isotropic non-functional materials. On the other hand, magnetoelectric (ME) solids are of particular interest, converting electrical to magnetic energy and vice versa. The constitutive behavior of those materials has been investigated extensively, however restricting considerations to static or quasi-static loading. The goal of this paper is to combine electrodynamics in terms of the classical Maxwell equations with the constitutive behavior of ME materials. The interacting mechanisms of ME energy conversion lead to a complex behavior and are supposed to give rise to interesting phenomena influencing wave dispersion, deflection and reflection. The coupled boundary value problem is comprehensively formulated first, including mechanical stress and strain fields as well as electromagnetically induced forces. Gauge transformations are presented to decouple the electrodynamic potential equations for anisotropic ME bodies, neglecting a mechanical compliance at that point. Weak formulations are derived as a basis for numerical discretization procedures like the finite element method and simple examples demonstrate the impact of ME coupling on the phase velocity of an electromagnetic wave.

1. Introduction

In a dielectric body, the conversion of electrical and magnetic energies can be accomplished in two ways. First, a transient magnetic flux induces an electric vortex field, vice versa a transient electric flux induces a magnetic vortex field. Second, the coupling can be due to the constitutive behavior of a dielectric. This magnetoelectric (ME) effect is also observed at the static limit of electric and magnetic fields and is rarely observed in materials natively. The few crystal classes intrinsically displaying the ME effect exhibit a weak coupling at very low temperatures [1]. In composite structures, however, the ME coupling can be observed at room temperature and its intensity is appropriate for technical applications, e.g. as highly sensitive magnetic field sensors or efficient data storage devices [2,3]. In ME composites, piezoelectric and magnetostrictive constituents

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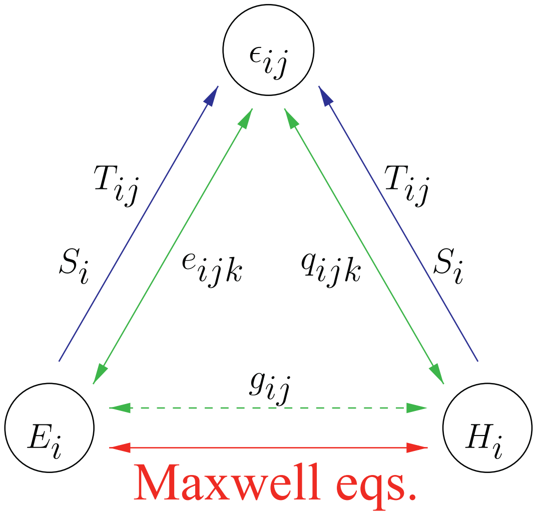

Fig. 1. Interactions between magnetic (H_i), electric (E_i) and mechanical (ϵ_{ij}) fields due to constitutive relations (q_{ijk}, e_{ijk}, g_{ij}), electromagnetic forces (T_{ij}, S_i) and Maxwell's equations (small strain assumption).

are combined on the nano-, micro- or macroscale [4–6]. The coupling of magnetic and electric fields is accomplished indirectly via strain fields due to the presence of both direct and inverse piezoelectric and piezomagnetic effects. Thus, from the mathematical point of view, a three-field problem has to be solved in terms of electrical, magnetic and mechanical fields.

The electromagnetic coupling depicted at the beginning of this section is described by the famous Maxwell equations of electrodynamics. Representing a compact system of partial differential equations, their solution is not straightforward, though. At least six unknown scalar quantities have to be determined in a space of electric and magnetic fields. A vector and a scalar potential can be substituted into the equations, leaving four unknowns. Quasi-static approximations within an electrodynamical context lead to further reductions of the problem, e.g. neglecting electric displacement rates, thus still keeping one vector and one scalar potential resulting, however, in a more compact system of differential equations. In the most simple and limiting case of pure electromagnetostatics two decoupled Laplacian differential equations with two unknown scalar potentials are obtained [7]. In this paper, the most general dynamic case of electromagnetic coupling in a dielectric medium at rest will be considered. Introducing two potentials substituting magnetic induction and electric field, several gauge transformations are known from literature [8]. The goal is always to simplify the solution procedure by completely or at least partially decoupling Maxwell's equations. In doing so, the constitutive behavior of the considered medium has to be specified. The gauge transformations, however, are hitherto only available for isotropic media such as vacuum or air.

Being a property of the material, the ME effect is mathematically described by constitutive equations. These are more complex than those for a non-functional dielectric, exhibiting coupling coefficients and in general showing anisotropic behavior. In a composite, the principle of ME energy conversion being based on the strain fields, the constitutive framework has to cover additional coupling mechanisms, i.e. piezoelectricity and magnetostriction. In general, the coefficients themselves are nonlinear functions of the independent magnetic, electric and mechanical variables and their time history, leading to hysteresis characteristics. Within a small signal range, however, the constitutive behavior can be assumed to be linear. The same holds for high-frequency loading e.g. due to electromagnetic waves.

Linear and nonlinear modeling of continuous ME systems is subject of various scientific papers [4,9–11]. Related boundary value problems are, to our best knowledge, always static or quasi-static within a low-frequency range. On the other hand, analytical and numerical methods in electrodynamics are quite advanced, however restricting applications to non-functional materials. A combination of both aspects of magnetoelectric energy conversion, i.e. within both a constitutive framework of ME materials and Maxwell's equations, is goal of this paper. Different aspects will have to be considered therefore. First, a comprehensive mathematical framework is set up, accounting for different interactions in between mechanical (ϵ_{ij}), electrical (E_i) and magnetic (H_i) fields. Fig. 1 illustrates the multiple interactions. The constitutive relations, indicated by the coefficients q_{ijk}, g_{ij} and e_{ijk}, are in principle connecting all three fields. In a native ME material the direct relation, which is marked with the dashed line, is exploited, however piezoelectric and magnetostrictive coupling may be observed as well. In a composite, the latter effects on the microscale lead to a ME coupling on the continuum or macro level. The Maxwell equations only connect electric and magnetic fields. The latter only holds for small strain assumptions, whereas finite strain leads to a coupling with mechanical fields due to the changing domain of the electrodynamic boundary value problem. This effect is essential for electroactive or magnetoactive polymer modeling which, however, is not within the scope of our considerations. Finally, there are forces and thus stresses induced by an electromagnetic field, acting on the boundary and in the domain of the dielectric. Here, Maxwell stresses (T_{ij}) and the Poynting vector (S_i) play important roles, which will be pointed out in the next section. This kind of coupling is just one-sided if finite deformations remain disregarded.