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Nonlinear Choquard equations: Doubly critical case

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ABSTRACT

Consider nonlinear Choquard equations

$$\begin{cases} -\Delta u + u = (I_\alpha * F(u))F'(u) & \text{in } \mathbb{R}^N, \\ \lim_{x\to\infty} u(x) = 0, \end{cases}$$

where I_α denotes Riesz potential and $\alpha \in (0, N)$. In this paper, we show that when F is doubly critical, i.e. $F(u) = \frac{N}{N+\alpha}|u|^{\frac{N+\alpha}{N}} + \frac{N-2}{N+\alpha}|u|^{\frac{N+\alpha}{N-2}}$, the nonlinear Choquard equation admits a nontrivial solution if $N \ge 5$ and $\alpha + 4 < N$.

1. Introduction

Let $N \ge 3$, $\alpha \in (0, N)$. We are concerned with the nonlinear Choquard equation:

$$\begin{cases} -\Delta u + u = (I_\alpha * F(u))F'(u) & \text{in } \mathbb{R}^N, \\ \lim_{x\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where I_α is Riesz potential given by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}}$$

and Γ denotes the Gamma function. It is the Euler–Lagrange equation of the functional

$$J_\alpha(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\,dx.$$

Physical motivation of (1.1) comes from the case that $F(u) = \frac{1}{2}|u|^2$ and $\alpha = 2$. In this case, Eq. (1.1) is called the Choquard–Pekar equation [1,2], Hartree equation [3,4] or Schrödinger–Newton equation [5,6], depending

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on its physical backgrounds and derivations. The existence of a ground state in this case is studied in [2,7,8] via variational arguments.

The functional J_α can be considered as a nonlocal perturbation of the fairly well-studied functional consisting of only local terms:

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\, dx - \int_{\mathbb{R}^N} G(u)\, dx$$

since as $\alpha \to 0$, J_α approaches to J_0 with $G(u) = \frac{1}{2}F^2(u)$. A critical point of J_0 is a solution to the stationary nonlinear Schrödinger equation:

$$-\Delta u + u = G'(u). \tag{1.2}$$

The power type function $\frac{1}{p}|u|^p$ is a standard choice for nonlinearity $G(u)$ (and also $F(u)$). By Sobolev inequality, it can be shown that the functional J_0 is a well-defined C^1 functional on $H^1(\mathbb{R}^N)$ if $G(u) = \frac{1}{p}|u|^p$ and $p \in [2, \frac{2N}{N-2}]$. It is a classical result that it admits a nontrivial critical point of ground state level in the subcritical range $p \in (2, \frac{2N}{N-2})$ [9,10]. Moreover, the standard application of Pohozaev's identity says that if p is out of subcritical, i.e., $1 < p \le 2$ or $p \ge 2N/(N-2)$, Eq. (1.2) does not admit any nontrivial finite energy solution. In case of J_α, Hardy–Littlewood–Sobolev inequality (Proposition 2.1 below) replaces Sobolev inequality to see that J_α with $F(u) = \frac{1}{p}|u|^p$ is well-defined and is continuously differentiable on $H^1(\mathbb{R}^N)$ if $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$. Two numbers $\frac{N+\alpha}{N}$ and $\frac{N+\alpha}{N-2}$ play roles of lower and upper critical exponents for existence. It is proved by Moroz and Van schaftingen [11] that for every $\alpha \in (0, N)$, there exists a nontrivial ground state solution if p is in the subcritical range, i.e., $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and there is no nontrivial finite energy solution if p is outside of subcritical, i.e., $1 < p \le \frac{N+\alpha}{N}$ or $p \ge \frac{N+\alpha}{N-2}$. This result is compatible with the existence of the limit equation (1.2). Observe the existence range $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ tends to $p \in (1, \frac{N}{N-2})$ and the nonlinear term $(I_\alpha * |u|^p)|u|^p$ tends to $|u|^{2p}$ as $\alpha \to 0$. We recall that Eq. (1.2) with $G(u) = \frac{1}{2}F^2(u) = \frac{1}{2p^2}|u|^{2p}$ admits a nontrivial finite energy solution if and only if $p \in (1, \frac{N}{N-2})$. Furthermore, we have H^1 convergence between ground states. For any $p \in (1, \frac{N}{N-2})$, choose a small $\alpha_0 > 0$ that p belongs to the segment $(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ for every $\alpha \in (0, \alpha_0)$ so that a radial positive ground state u_α to (1.1) with $F(u) = \frac{1}{p}|u|^p$ exists. Then it is possible to show that as $\alpha \to 0$, u_α converges in H^1 sense to a ground state u_0 of the corresponding functional J_0. See [12,13].

For general nonlinearity G, Berestycki and Lions prove in their celebrated paper [9] that (1.2) admits a ground state solution when G is $C^1(\mathbb{R})$ and satisfies the following:

(G1) there exists a constant $C > 0$ such that for every $s \in \mathbb{R}$,

$$|sG'(s)| \le C(|s|^2 + |s|^{\frac{2N}{N-2}}),$$

(G2) $\lim_{s\to\infty} \frac{G(s)}{|s|^{\frac{2N}{N-2}}} = 0$ and $\lim_{s\to 0} \frac{G(s)}{|s|^2} = 0$,

(G3) there exists a constant $s_0 \in \mathbb{R} \setminus \{0\}$ such that $G(s_0) > \frac{s_0^2}{2}$.

In the same spirit, it is proved in [14] that there exists a ground state solution to (1.1) under the following conditions for the nonlinearity function $F \in C^1(\mathbb{R})$:

(F1) (growth) there exists a constant $C > 0$ such that for every $s \in \mathbb{R}$,

$$|sF'(s)| \le C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}),$$

(F2) (subcriticality) $\lim_{s\to\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0$ and $\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0$,

(F3) (nontriviality) there exists a constant $s_0 \in \mathbb{R} \setminus \{0\}$ such that $F(s_0) \ne 0$.