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On the uniqueness of the limit cycle for the Liénard equation with $f(x)$ not sign-definite

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A B S T R A C T

The problem of uniqueness of limit cycles for the Liénard equation $\ddot{x}+f(x)\dot{x}+g(x) = 0$ is investigated. The classical assumption of sign-definiteness of $f(x)$ is relaxed. The effectiveness of our result as a perturbation technique is illustrated by some constructive examples of small amplitude limit cycles, coming from bifurcation theory.

1. Introduction and discussion about some uniqueness results

The aim of this paper is to investigate the problem of uniqueness of the limit cycle for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous function and $g(x)$ is locally Lipschitz and satisfies the sign condition $g(x)x > 0$ for $x \neq 0$.

The first result in this direction was actually achieved by Liénard himself in his pioneering paper [1] which is still a milestone in this area. Observe that in the same paper the Liénard plane

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{2}$$

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was introduced, where $F(x) = \int_0^x f(s)\,ds$. It is well known that the study of Eq. (1) in such a plane is equivalent to the study in the phase-plane

$$\dot{x} = y \quad \dot{y} = -f(x)y - g(x).$$

Liénard [1] proved the uniqueness of the limit cycle under the following assumption

($\mathscr{F}$) $f(0) < 0$, $F(x)$ has precisely three zeros $\alpha < 0 < \beta$, and is monotone increasing outside the interval $[\alpha, \beta]$.

It should be observed that Liénard was treating the case $f(x)$ even (and therefore $F(x)$ odd) and $g(x) = x$. This fact produces some obvious symmetries when studying (2) and is useful in the proof because it gives the property:

($\mathscr{S}$) *All the possible limit cycles cross both the lines $x = \alpha$ and $x = \beta$*

(clearly, in the case studied by Liénard it is $\alpha = -\beta$). Such uniqueness result was then improved by Levinson and Smith [2] and by Sansone [3], still keeping the symmetry property. More in detail Levinson and Smith [2] assumed $f(x)$ even and $g(x)$ odd, while Sansone [3] considered the case $g(x) = x$ and $F(\alpha) = F(\beta)$. As a consequence, in both the cases the property ($\mathscr{S}$) is satisfied. Such a property plays a crucial role as it was shown in the classical counterexample by Duff and Levinson [4]. Indeed, in [4] the authors produce an example in which $F(x)$ satisfies ($\mathscr{F}$), $g(x) = x$ but three limit cycles are present, two of them not intersecting one of the vertical lines. For this reason in the search of hypotheses ensuring the uniqueness of the limit cycle and in order to avoid the symmetry property $F(x)$ and $g(x)$ odd, one has to impose condition ($\mathscr{S}$). This fact was well known, but to our knowledge it was explicitly stated by Roberto Conti in [5] in his Italian notes for an advanced course in ODEs. Restating this result in our context, we have in fact the following:

Theorem 1.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ as above. Assume that there exist α, β with $\alpha < 0 < \beta$ such that*

($\mathscr{F}_1$) *$F(\alpha) = F(\beta) = 0$ with $F(x)x < 0$ for $x \in]\alpha, \beta[$, $x \neq 0$ and $F(x)x > 0$ for $x < \alpha$ and for $x > \beta$;*
($\mathscr{F}_2$) *$F(x)$ is monotone increasing for $x < \alpha$ and for $x > \beta$.*

Then system (2) has at most one limit cycle provided that property ($\mathscr{S}$) holds.

Notice that ($\mathscr{F}$) implies ($\mathscr{F}_1$) and ($\mathscr{F}_2$). Moreover, in Theorem 1.1 no symmetry condition on f or g is required. On the other hand, to verify assumption ($\mathscr{S}$) is not an easy task. In this light, an elegant and easy verifiable condition for ($\mathscr{S}$) is given by $G(\alpha) = G(\beta)$ (cf., for instance, [6]). Results depending on Theorem 1.1 were explicitly or implicitly used by several researchers with the main goal of producing sufficient conditions for the property ($\mathscr{S}$) and hence the uniqueness (see [6–14]).

In the study of this problem, Lefschetz in his classical book (still in the framework of the symmetry conditions) [15, p. 272, Fig. 2] observed that if the monotonicity property of $F(x)$ outside the interval $[\alpha, \beta]$ is omitted, there is the possibility of *giving rise to a succession of "concentric" closed paths $\Gamma_1, \Gamma_2, \ldots$ and Γ_i will be orbitally stable (unstable)* alternatively. Perhaps in view of this remark, surprisingly little attention has been paid to relax the monotonicity hypothesis in this problem. Therefore, all the above quoted results must assume $f(x)$ to be positive for $|x|$ large.

In the present paper we attack the uniqueness problem for Eq. (1), by relaxing the monotonicity assumption ($\mathscr{F}_2$) on $F(x)$. More precisely, we will give evidence of the fact that uniqueness of the limit cycle can be guaranteed for a wide range of cases in which $F(x)$, as well as $f(x)$ oscillates or is eventually negative. Constructive examples will be presented. Throughout the article when speaking of uniqueness of limit cycle, we mean that we consider the fact that *at most* one limit cycle does exist and we do not focus