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The boundness of the operator-valued functions for multidimensional nonlinear wave equations with applications

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ABSTRACT

The operator-variation-of-constants formula was derived by Wu et al. (2015) for the general multidimensional nonlinear wave equations, and the authors proved that the formula is adapted to different boundary conditions. Furthermore, an energy-preserving scheme for one-dimensional (1D) nonlinear Hamiltonian wave equations with periodic boundary conditions was proposed by Liu et al. (2016). It is known that the formula is associated with the operator-valued functions which depend on Laplacian. Hence, it is crucial to show the boundness of the operator-valued functions. This motivates the new study of the boundness of the operator-valued functions. As an application, we extend the energy-preserving scheme from 1D to multidimensional nonlinear Hamiltonian wave equations with three different boundary conditions.

1. Introduction

In this letter, we investigate the following multidimensional nonlinear wave equations

$$\begin{cases} u_{tt} - a^2\Delta u = f(u), & x \in \mathbf{\Omega},\ t_0 < t \leq T, \\ u(x,t_0) = \varphi_0(x), \quad u_t(x,t_0) = \varphi_1(x), & x \in \bar{\mathbf{\Omega}}, \end{cases} \tag{1}$$

where $\mathbf{\Omega} = (0, L_1) \times \cdots \times (0, L_d) \subset \mathbb{R}^d$ is a rectangular domain and $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$. Three kinds of boundary conditions can be supplemented to these problems over the domain $\bar{\mathbf{\Omega}} = [0, L_1] \times \cdots \times [0, L_d] \subset \mathbb{R}^d$, which are listed as follows:

(i) *Periodic Boundary Conditions:* $u(t,x)|_{\partial\mathbf{\Omega}\cap\{x_j=0\}} = u(t,x)|_{\partial\mathbf{\Omega}\cap\{x_j=L_j\}},\ j = 1, 2, \ldots, d;$
(ii) *Dirichlet Boundary Conditions:* $u|_{\partial\mathbf{\Omega}} = 0;$
(iii) *Neumann Boundary Conditions:* $\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial\mathbf{\Omega}} = 0.$

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Before analysing the boundness of the operator-valued functions, we define the linear differential operator $\mathcal{A}$ as

$$(\mathcal{A}v)(x) = -a^2\Delta v(x). \tag{2}$$

Obviously, $\mathcal{A}$ is an unbounded symmetric positive semi-definite operator and not defined for every $v \in L^2(\mathbf{\Omega})$. According to the specified boundary condition, the linear differential operator $\mathcal{A}$ can be defined on a suitable functional space $D(\mathcal{A}) = \Big\{u \in H^\sigma(\mathbf{\Omega}),\ u \text{ satisfies a suitable boundary condition}\Big\}$, where σ depends on the operator $\mathcal{A}$ and the corresponding boundary condition. In the very recent papers (see, e.g. [1–4]), by defining $u(t)$ as the function that maps x to $u(t,x)$: $u(t) = [x \mapsto u(t,x)]$, the authors formulated the original systems as the following abstract ordinary differential equations:

$$\begin{cases} u''(t) + \mathcal{A}u(t) = f\big(u(t)\big), & 0 < t \le T, \\ u(t_0) = \varphi_1(x), & u'(t_0) = \varphi_2(x), \end{cases} \tag{3}$$

on the closed subspace $\mathcal{X} := \Big\{u(\cdot,x) \in X \ \Big|\ u(\cdot,x) \text{ satisfies the corresponding boundary conditions}\Big\} \subseteq L^2(\mathbf{\Omega})$, where $X = \text{lin}\{w_k(x) : k \in \mathcal{I}^d\}$ is dense in the Hilbert space $L^2(\mathbf{\Omega})$ and all $w_k(x)$ are the orthonormal eigenvectors of the operator $\mathcal{A}$. The operator-variation-of-constants formula for the initial-value problem of the general multidimensional nonlinear wave equation (3) has been established as follows.

Theorem 1.1. *The solution of the abstract ODE (3) and its derivative satisfy*

$$\begin{cases} u(t) = \phi_0\big((t-t_0)^2\mathcal{A}\big)u(t_0) + (t-t_0)\phi_1\big((t-t_0)^2\mathcal{A}\big)u'(t_0) + \displaystyle\int_{t_0}^{t}(t-\zeta)\phi_1\big((t-\zeta)^2\mathcal{A}\big)f\big(u(\zeta)\big)\mathrm{d}\zeta, \\ u'(t) = -(t-t_0)\mathcal{A}\phi_1\big((t-t_0)^2\mathcal{A}\big)u(t_0) + \phi_0\big((t-t_0)^2\mathcal{A}\big)u'(t_0) + \displaystyle\int_{t_0}^{t}\phi_0\big((t-\zeta)^2\mathcal{A}\big)f\big(u(\zeta)\big)\mathrm{d}\zeta, \end{cases} \tag{4}$$

for $t \ge t_0$, where $\phi_0\big((t-t_0)^2\mathcal{A}\big)$ and $\phi_1\big((t-t_0)^2\mathcal{A}\big)$ are functions of the differential operator $\mathcal{A}$.

Furthermore, the authors showed that the established operator-variation-of-constants formula is adapted to different boundary conditions [2], developed the energy-preserving and symmetric scheme for 1D nonlinear Hamiltonian wave equation [1], and proposed the semi-analytical explicit integrator for PDEs [3]. On the other hand, we are also aware of an important fact that the operator-valued functions $\phi_j(\mathcal{A})$ for $j = 0, 1$ are required in the application of the formula (4) and the relevant discussion. Moreover, the boundness of the operator-valued functions is essential once numerical approximations are considered since the operator-valued functions are involved in the definite integrals. Thus, in this letter, we will pay our attention to show the boundness and some useful properties of the operator-valued functions.

It should be noted that once the nonlinear wave equations expressed by (1) are Hamiltonian systems with $f(u) = -\nabla V(u)$, i.e., Klein–Gordon equations, the following energy conservation law becomes a key feature:

$$E(t) = \frac{1}{2}\int_{\mathbf{\Omega}}\big(u_t^2 + a^2|\nabla u|^2 + 2V(u)\big)\mathrm{d}x \equiv E(t_0). \tag{5}$$

In the paper [1], the authors derived and analysed a novel energy-preserving and symmetric scheme for 1D nonlinear Hamiltonian wave equations (1) with periodic boundary conditions. As a development of the derived energy-preserving scheme, we will prove that it is also suitable to the multidimensional nonlinear Hamiltonian wave equations equipped with three different kinds of boundary conditions.

The rest of this letter is organised as follows. In Section 2, we analyse the boundness and some useful properties of the operator-valued functions $\phi_j(\mathcal{A})$, $j \in \mathbb{N}$ for *Periodic*, *Dirichlet* and *Neumann* boundary conditions over a multidimensional rectangular domain. In Section 3, the energy-preserving scheme is extended to multidimensional problems. Section 4 is devoted to the conclusion.