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On the uniqueness of solutions for a class of fractional differential equations ☆

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ABSTRACT

We are concerned with the uniqueness of solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} D^p x(t) + f(t, x(t)) = 0, \ 2 < p \leq 3, \ t \in (0,1), \\ x(0) = x'(0) = 0, \quad x(1) = 0 \end{cases}$$

where D^p_{0+} denotes the standard Riemann–Liouville fractional derivative. Our analysis relies on the theory of linear operators and the $\| \cdot \|_e$ norm.

1. Introduction

The boundary value problems for fractional differential equations arise from the studies of models of aerodynamics, fluid flows, electrodynamics of complex medium, electrical networks, rheology, polymer rheology, economics, biology chemical physics, control theory, signal and image processing. Recently, the study of such kind of problems has received considerable attention both in theory and applications, see [1–13] and the references therein. In particular, existence, uniqueness and qualitative analysis of solutions of boundary value problems for fractional differential equations have been considered by many authors, see [14–19]. For example, cui [14] considered the following boundary value problems for fractional differential equations

$$\begin{cases} D^p x(t) + p(t) f(t, x(t)) + q(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \quad x(1) = 0 \end{cases}$$

where D^p_{0+} is the standard Riemann–Liouville derivative, $2 < p \leq 3$ is a real number, $q : (0,1) \to [0, +\infty)$ is Lebesgue integrable and does not vanish identically on any subinterval of $(0,1)$. By use of u_0-positive

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operator, he obtained the uniqueness existence of solutions for the above fractional differential equation under the assumption that $f(t,x)$ is a Lipschitz continuous function. It should be mentioned that the Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators, and therefore the Lipschitz condition is optimal. However, it is very difficult to compute the value of the first eigenvalues even for boundary value problems for integer differential equation.

In this paper, we devote to the investigation of the uniqueness of solutions of fractional differential equations

$$\begin{cases} D^p x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \quad x(1) = 0, \end{cases} \tag{1.1}$$

where $2 < p \leq 3$ is a real number. Namely, we prove that a general theorem on the uniqueness of solutions to (1.1) under a mild Lipschitz condition on f (see Theorem 3.1) by use of e-norm. We should address here that our work presented in this paper has various new features. First of all, the Lipschitz condition is easy to verify. Second, our study is on singular nonlinear differential boundary value problems, that is, $f(t,x)$ may be singular at $t = 0, 1$.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and give some useful preliminary results. In Section 3, we show the uniqueness of solutions to (1.1).

2. Preliminaries

In this section, we introduce notions, definitions, and preliminary results which are used throughout this paper. By $C[0,1]$, we denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ with the norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Definition 2.1 ([1]). The Riemann–Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{p-n+1}} ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([10]). *Let $\sigma \in C(0,1) \cap L(0,1)$ and $2 < p \leq 3$, then the unique solution of*

$$\begin{cases} D^p x(t) + \sigma(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \quad x(1) = 0, \end{cases}$$

is given by

$$x(t) = \int_0^1 G(t,s)\sigma(s) ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{p-1}t^{p-1} - (t-s)^{p-1}}{\Gamma(p)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(1-s)^{p-1}t^{p-1}}{\Gamma(p)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

Lemma 2.2 ([10]). *The function $G(t,s)$ defined by (2.1) satisfies the following conditions:*

(1) $t^{p-1}(1-t)s(1-s)^{p-1} \leq \Gamma(p)G(t,s) \leq (p-1)s(1-s)^{p-1},\ t,s \in (0,1)$,

(2) $t^{p-1}(1-t)s(1-s)^{p-1} \leq \Gamma(p)G(t,s) \leq (p-1)t^{p-1}(1-t),\ t,s \in (0,1)$.