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On a high-order one-parameter family for the simultaneous determination of polynomial roots

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A B S T R A C T

A new one-parameter family of simultaneous methods for the determination of all (simple or multiple) zeros of a polynomial is derived. The order of the basic family of simultaneous methods is four. Using suitable corrective approximations, the order of convergence of this family is increased even to six without any additional evaluations of the polynomial and its derivatives, which points to a high computational efficiency of the new family of root solvers. Comparison with the existing methods in regard to computational efficiency and numerical examples are also performed.

1. One-parameter family of simultaneous methods

The aim of this paper is to derive a new one-parameter family of simultaneous methods for the determination of all zeros of a polynomial. This method is based on a one-parameter cubically convergent method for finding a multiple zero of at least two-times differentiable function f with a zero α of the known multiplicity $m \geq 1$ presented in [1]. The mentioned third order method for an isolated multiple zero is given by the iterative formula

$$\hat{x} = x - \frac{mu(x)(2+\beta u(x))}{1+m+\big(\beta - 2mA_2(x)\big)u(x)}, \tag{1}$$

where β is a real parameter, $\hat{x}$ is a new approximation and $u(x) = \frac{f(x)}{f'(x)}$, $\quad A_2(x) = \frac{f''(x)}{2f'(x)}$.

Let P be a monic polynomial of degree N with simple or multiple zeros $\alpha_1, \ldots, \alpha_n$ $(n \leq N)$ of the respective multiplicities $m_1, \ldots, m_n$ known in advance. Let us introduce

$$\delta_{\lambda,i} = \frac{P^{(\lambda)}(x_i)}{P(x_i)}, \quad S_{\lambda,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{(x_i - x_{j,h})^{\lambda}}, \quad d_i = \delta_{2,i} - \delta_{1,i}^2 + S_{2,i} \quad (\lambda = 1, 2),$$

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where x_i are some approximations to α_i and $x_{j,h}$ are Halley-like *corrective approximations* defined by the third order iterative formula [2]

$$x_{j,h} = x_j - \frac{2\delta_{j,1}}{\frac{m_j+1}{m_j}\delta_{j,1}^2 - \delta_{j,2}}. \tag{2}$$

Let us define rational function

$$W_i(x) = \frac{P(x)}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (x - x_{j,h})^{m_j}} \quad (x \neq x_{j,h},\ i \in \{1,\ldots,n\}). \tag{3}$$

Obviously, the function $W_i(x)$ has the same zeros as the polynomial $P(x)$. Starting from (3) and applying the logarithmic derivatives, we find

$$\frac{W_i'(x_i)}{W_i(x_i)} = \delta_{1,i} - S_{1,i} =: r_{1,i}, \quad \frac{W_i''(x_i)}{W_i'(x_i)} = \delta_{1,i} - S_{1,i} + \frac{\delta_{2,i} - \delta_{1,i}^2 + S_{2,i}}{\delta_{1,i} - S_{1,i}} =: r_{2,i} = r_{1,i} + \frac{d_i}{r_{1,i}}. \tag{4}$$

Note that Ehrlich applied the rational function (3) to Newton's method to derive the third order simultaneous method, see [3].

Substitute

$$u(x_i) = \frac{P(x_i)}{P'(x_i)} \ \text{ with } \ \frac{W_i(x_i)}{W_i'(x_i)} \ \text{ and } \ A_2(x_i) = \frac{P''(x_i)}{2P'(x_i)} \ \text{ with } \ \frac{W_i''(x_i)}{2W_i'(x_i)}$$

in (1). Using the expressions (4) and the abbreviation $\rho_i = r_{1,i} + \beta$, we obtain the new one-parameter family of iterative methods for the simultaneous determination of multiple zeros of a polynomial

$$\hat{x}_i = x_i - \frac{m_i(r_1 + \rho_i)}{r_1\rho_i - m_i d_i} \quad (i = 1,\ldots,n). \tag{5}$$

Introducing the iteration index k, we present the family (5) in the form

$$x_i^{(k+1)} = x_i^{(k)} - \frac{m_i\big(r_{1,i}^{(k)} + \rho_i^{(k)}\big)}{r_{1,i}^{(k)}\rho_i^{(k)} - m_i d_i^{(k)}}, \quad \rho_i^{(k)} = r_{1,i}^{(k)} + \beta, \quad (i = 1,\ldots,n,\ k = 0,1,\ldots). \tag{6}$$

In the next section, we will prove that the method (6) reaches the order six without any additional evaluations of the polynomial P and its derivatives. This means that the family (6) possesses a high computational efficiency. This preference and the possibility to produce a variety of simultaneous methods are the main advantages of the newly constructed family (6) of iterative methods.

2. Convergence analysis

In what follows, we will omit the iteration index k for brevity and consider the iterative formula (5) in the convergence analysis rather than (6). Let us introduce the errors $\varepsilon_i = x_i - \alpha_i$, $\varepsilon_{i,h} = x_{i,h} - \alpha_i$, $\hat{\varepsilon}_i = \hat{x}_i - \alpha_i$. Let $|\varepsilon|$ be the absolute value of the error ε of maximal magnitude, $|\varepsilon| = \max_{1\le j\le n} |\varepsilon_j|$. Assume that magnitudes of all errors $\varepsilon_1,\ldots\varepsilon_n$ are approximately of the same order, then $\varepsilon_j = O_M(\varepsilon_i)$ and $\varepsilon_j = O_M(\varepsilon)$, where the notion $a = O_M(b)$ means that real or complex numbers a and b are of the same magnitude.

Theorem 1. *Assume that initial approximations $x_1^{(0)},\ldots,x_n^{(0)}$ are sufficiently close to the respective zeros $\alpha_1,\ldots,\alpha_n$ of a given polynomial P. Then, the order of convergence of the family of simultaneous methods (6) is equal to six.*