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Remarks on Bernoulli constants, gauge conditions and phase velocities in the context of water waves

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ABSTRACT

This short note is about the gauge condition for the velocity potential, the definitions of the Bernoulli constant and of the velocity speeds in the context of water waves. These definitions are often implicit and thus the source of confusion in the literature. This note aims at addressing this issue. The discussion is related to water waves because the confusion are frequent in this field, but it is relevant for more general problems in fluid mechanics.

1. Introduction

The Euler equations describe the momentum conservation of an inviscid fluid. For irrotational motions of incompressible fluids, the Euler equations can be integrated into a scalar equation called *Bernoulli equation* for steady flows and *Cauchy–Lagrange equation* for unsteady flows. The Bernoulli and Cauchy–Lagrange equations resulting of an integration procedure, they involve an arbitrary integration 'constant', the so-called *Bernoulli constant* (that is actually an arbitrary function of time for the Cauchy–Lagrange equation). This Bernoulli 'constant' and its physical meaning is a frequent source of confusion in the literature, especially in the study of water waves. Thus, there has been some recent works aiming at clarifying the situation [1].

The purpose of this short note is to address the issues related to the Bernoulli constants. This leads to clarify the definition of the velocity potential, its uniqueness being introduced by a gauge condition, and how this quantity is modified via Galilean transformations. Various frames of references are also discussed as they lead to the definition the phase velocity of a wave.

2. Cauchy–Lagrange equation

For the sake of simplicity, we consider the two-dimensional motion of an homogeneous incompressible fluid, but this is not a limitation for the purpose of the present note. For inviscid fluids, the equations of

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motion are [2]

$$u_x \;+\; u_y \;=\; 0, \tag{1}$$

$$u_t \;+\; u\,u_x \;+\; v\,u_y \;=\; -\,p_x, \tag{2}$$

$$v_t \;+\; u\,v_x \;+\; v\,v_y \;=\; -\,p_y \;-\; g, \tag{3}$$

where $\boldsymbol{x} = (x, y)$ are the Cartesian coordinates (y being directed upward), t is the time, $\boldsymbol{u} = (u, v)$ is the velocity field, p is the pressure divided by the (constant) density and g is the (constant) acceleration due to gravity directed toward the decreasing y-direction (downward).

In irrotational motion $v_x = u_y$, so there exists a velocity potential ϕ such that $u = \phi_x$ and $v = \phi_y$, i.e. $\boldsymbol{u} = \operatorname{grad}\phi$. The Euler equations (2)–(3) can then be rewritten [2]

$$\operatorname{grad}\left[\,\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p\,\right] \;=\; 0, \tag{4}$$

that can be integrated into the Cauchy–Lagrange equation

$$\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p \;=\; C(t), \tag{5}$$

where C is an integration 'constant' often called *Bernoulli constant* or *Bernoulli integral*.

3. Gauge condition

The velocity potential being defined via its gradient, ϕ is not an unique function: adding any arbitrary function of time to ϕ does not change the velocity field. Thus, if one makes the change of potential [2]

$$\phi(\boldsymbol{x},t) \;=\; \phi^\star(\boldsymbol{x},t) \;+\; \int C(t)\,\mathrm{d}t,$$

so that $\operatorname{grad}\phi = \operatorname{grad}\phi^\star$, the Cauchy–Lagrange equation (5) becomes

$$\phi^\star_t \;+\; \frac{1}{2}\,(\phi^\star_x)^2 \;+\; \frac{1}{2}\,(\phi^\star_y)^2 \;+\; g\,y \;+\; p \;=\; 0. \tag{6}$$

In other words, this shows that it is always possible, via a suitable definition of the velocity potential, to take

$$C(t) \;=\; 0, \tag{7}$$

without loss of generality and preserving the velocity field (i.e., $\operatorname{grad}\phi = \boldsymbol{u}$). Enforcing the unicity of the velocity potential ϕ (up to an additional constant) via (7) is a so-called *gauge condition*.

Hereafter, we always take the gauge condition (7) and the Cauchy–Lagrange equation is thus

$$\phi_t \;+\; \frac{1}{2}\,(\phi_x)^2 \;+\; \frac{1}{2}\,(\phi_y)^2 \;+\; g\,y \;+\; p \;=\; 0. \tag{8}$$

Of course, other gauge conditions could be introduced, as well as no gauge condition at all. In the latter case, the arbitrary function $C(t)$ should be carried along all the derivations.

Note that with the gauge (7), the Bernoulli 'constant' disappears from the Cauchy–Lagrange equation (8), but it has not completely been eliminated: it is now 'hidden' in the definition of the velocity potential ϕ and will reappear explicitly for some special flows, as shown below.