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Preprocessing schemes for fractional-derivative problems to improve their convergence rates

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A simple and inexpensive preprocessing of an initial–boundary value problem with a Caputo time derivative is shown theoretically and numerically to yield an enhanced convergence rate for the L1 scheme. The same preprocessing can also be used with other methods for time-dependent problems.

1. Introduction

Consider the initial–boundary value problem

$$Lu(x,t) := D_t^\alpha u - p u_{xx} + c(x)u = f(x,t) \tag{1a}$$

for $(x,t) \in Q := (0,l) \times (0,T]$, with

$$u(0,t) = u(l,t) = 0 \quad \text{for } t \in (0,T], \tag{1b}$$

$$u(x,0) = \phi(x) \quad \text{for } x \in [0,l], \tag{1c}$$

where $0 < \alpha < 1$, p is a positive constant, $c \in C[0,l]$ with $c \geq 0$, $f \in C(\bar{Q})$ where $\bar{Q} := [0,l] \times [0,T]$, and $\phi \in C[0,l]$. In (1a), $D_t^\alpha u$ denotes a Caputo fractional derivative of order α with $0 < \alpha < 1$, which is defined by

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t} (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s}\, ds \text{ for } (x,t) \in Q.$$

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It is known [1,2] that under reasonable hypotheses on its data, the problem (1) has a unique solution which typically exhibits a weak singularity at $t = 0$; roughly speaking, $u(x,t)$ behaves like the function t^α near $t = 0$. Consequently the temporal derivatives u_t and u_{tt} are unbounded at $t = 0$, which presents challenges for numerical methods for (1), and for their analysis. Many different numerical methods have been suggested for (1); see [3] for a list of these.

In the present paper we show that a simple preprocessing of the solution u of (1) will improve the rate of convergence of the computed solution when the well-known L1 scheme is used to approximate the fractional derivative $D_t^\alpha u$. While this scheme is the only one that we discuss here, nevertheless the preprocessing idea discussed in this paper can clearly be used to improve the rate of convergence of many methods for (1) and for other problems such as the fractional wave-diffusion equation (for which $1 < \alpha < 2$ in (1a)).

Notation. By C we denote a generic constant that depends on the data of the problem (1), i.e., $C = C(\alpha, p, c, f, \phi, l, T)$. It can take different values in different places but is independent of any mesh used.

2. The basic idea

As in [1], we set

$$D(\mathcal{L}^\gamma) = \left\{ \omega \in L_2(0,l) : \sum_{i=1}^{\infty} \lambda_i^{2\gamma} \left| \int_{s=0}^{l} \omega(s)\psi_i(s)\, ds \right|^2 < \infty \right\} \quad \text{for } \gamma \geq 0,$$

where $\{(\lambda_i, \psi_i) : i = 1, 2, \ldots\}$ are the eigenvalues and normalised eigenfunctions of the Sturm–Liouville two-point boundary value problem

$$\mathcal{L}\psi_i := -p\psi_i'' + c\psi_i = \lambda_i \psi_i \text{ on } (0,l), \quad \psi_i(0) = \psi_i(l) = 0. \tag{2}$$

The norm associated with the space $D(\mathcal{L}^\gamma)$ is denoted by $\|\cdot\|_{\mathcal{L}^\gamma}$.

As in [2, Theorem 2.1], assume that $\phi \in D(\mathcal{L}^{5/2})$, $f(\cdot,t) \in D(\mathcal{L}^{5/2})$, $f_t(\cdot,t)$ and $f_{tt}(\cdot,t)$ are in $D(\mathcal{L}^{1/2})$ for each $t \in (0,T]$ with

$$\|f(\cdot,t)\|_{\mathcal{L}^{5/2}} + \|f_t(\cdot,t)\|_{\mathcal{L}^{1/2}} + t^\rho \|f_{tt}(\cdot,t)\|_{\mathcal{L}^{1/2}} \leq C_1$$

for all $t \in (0,T]$ and some constant $\rho < 1$, where C_1 is a constant independent of t. These hypotheses imply in particular that

$$0 = \phi(0) = \phi''(0) = \phi(l) = \phi''(l) = f(0,t) = f(l,t) \text{ for } 0 \leq t \leq T. \tag{3}$$

To solve (1) numerically, use the rectangular grid (x_m, t_n) where the uniform spatial mesh is $x_m = mh$ for $m = 0, 1, \ldots, M$, where $h = l/M$ for some positive integer M, while the temporal mesh is graded: $t_n = T(n/N)^r$ for $n = 0, 1, \ldots, N$ and the mesh grading $r \geq 1$ is chosen by the user. As in [2], we approximate $D_t^\alpha u$ by the L1 scheme and $-pu_{xx} + c(x)u$ by a standard $\mathcal{O}(h^2)$ finite difference approximation; we shall refer to this time and space discretisation simply as "the L1 scheme". It is shown in [2, Theorem 5.2] that the solution $\{u_m^n\}$ of (1) of this scheme satisfies

$$\max_{(x_m,t_n)\in\bar{Q}} |u(x_m,t_n) - u_m^n| \leq C\left(h^2 + N^{-\min\{2-\alpha,\, r\alpha\}}\right) \tag{4}$$

for some constant C.

In [4, Lemma 1] it is assumed that $\phi \in C^4[0,l]$, (3) holds true, $c \in C^2[0,l]$ and $f, f_x, f_{xx} \in C(\bar{Q})$, and it is shown that the solution u of (1) can be decomposed as

$$u(x,t) = z(x)t^\alpha + \phi(x) + v(x,t) \text{ for } (x,t) \in \bar{Q}, \tag{5}$$