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Oscillation of a linear delay differential equation with slowly varying coefficient

Mihály Pituk

Department of Mathematics, University of Pannonia, P.O. Box 158, 8201 Veszprém, Hungary

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ABSTRACT

A new sufficient condition is given for the oscillation of all solutions of a linear scalar delay differential equation with slowly varying and uniformly positive coefficient at infinity. Based on the celebrated Myshkis type necessary condition, we show that this new sufficient condition is sharp within the considered class of delay differential equations.

1. Introduction and the main result

Let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers and the set of nonnegative integers, respectively. Consider the linear delay differential equation

$$x'(t) = -p(t)x(t-\tau), \tag{1.1}$$

where

$$p \in C([t_0,\infty),\mathbb{R}^+) \qquad \text{and} \qquad \tau > 0. \tag{1.2}$$

The symbol $C([t_0,\infty),\mathbb{R}^+)$ denotes the set of nonnegative continuous functions on $[t_0,\infty)$.

By a solution of (1.1), we mean a function x which is continuous on $[t_1-\tau,\infty)$ for some $t_1 \geq t_0$, differentiable on $[t_1,\infty)$ and satisfies (1.1) for $t \geq t_1$. (By the derivative at $t = t_1$, we mean the right-hand side derivative.) Such a solution x of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*. Thus, x is nonoscillatory if it is eventually positive or eventually negative.

Define

$$\alpha = \liminf_{t\to\infty} p(t) \tag{1.3}$$

E-mail address: pitukm@almos.uni-pannon.hu.

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and

$$A = \limsup_{t\to\infty} p(t). \tag{1.4}$$

According to a result due to Myshkis [1], if

$$\alpha\tau > \frac{1}{e}, \tag{1.5}$$

then all solutions of (1.1) are oscillatory, and

$$A\tau < \frac{1}{e} \tag{1.6}$$

implies the existence of a nonoscillatory solution of (1.1). Thus, for the oscillation of all solutions of (1.1) it is necessary that

$$A\tau \ge \frac{1}{e}. \tag{1.7}$$

We will consider Eq. (1.1) in the case when coefficient p is slowly varying at infinity. Recall [2] that a function $p : [t_0,\infty) \to \mathbb{R}$ is *slowly varying at infinity* if for every $s \in \mathbb{R}$,

$$p(t+s) - p(t) \to 0 \qquad \text{as } t \to \infty. \tag{1.8}$$

Note that our terminology is different from the one used in the monograph by Seneta [3], where slowly varying functions are defined in multiplicative form. For the relationship (transformation) between the two different notions, see the remark below Theorem 1.2 in Chap. 1 of [3]. It follows from Lemma 1.4 in Chap. 1 of [3] that a Lebesgue measurable function $p : [t_0,\infty) \to \mathbb{R}$ is slowly varying at infinity if and only if there exists $t_1 \ge t_0$ such that p can be written in the form

$$p(t) = c(t) + \int_{t_1}^{t} \epsilon(s)\,ds, \qquad t \ge t_1, \tag{1.9}$$

where $c : [t_1,\infty) \to \mathbb{R}$ is a locally bounded Lebesgue measurable function such that

$$c(t) \to \gamma \qquad \text{for some } \gamma \in \mathbb{R} \text{ as } t \to \infty \tag{1.10}$$

and $\epsilon : [t_1,\infty) \to \mathbb{R}$ is a continuous function satisfying

$$\epsilon(t) \to 0 \qquad \text{as } t \to \infty. \tag{1.11}$$

As a simple consequence of the representation (1.9), we obtain that a continuous function $p : [t_0,\infty) \to \mathbb{R}$ is slowly varying at infinity if and only if there exists $t_1 \ge t_0$ such that p can be written in the form

$$p(t) = c(t) + d(t), \qquad t \ge t_1, \tag{1.12}$$

where $c : [t_1,\infty) \to \mathbb{R}$ is a continuous function which tends to a finite limit (1.10) and $d : [t_1,\infty) \to \mathbb{R}$ is a continuously differentiable function with property

$$d'(t) \to 0 \qquad \text{as } t \to \infty. \tag{1.13}$$

Our aim in this paper is to show that if p in Eq. (1.1) is slowly varying and uniformly positive at infinity in the sense that

$$\alpha > 0, \tag{1.14}$$

then (1.7) with a strict inequality is sufficient for the oscillation of all solutions of (1.1). Our main result is the following theorem.