Applied Mathematics Letters 73 (2017) 49–55

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Reconstructing a second-order Sturm–Liouville operator by an energetic boundary function iterative method

Chein-Shan Liu[a,b], Botong Li[c,*]

[a] Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu 210098, China
[b] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung 202-24, Taiwan
[c] School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

ARTICLE INFO

Article history:
Received 2 March 2017
Received in revised form 20 April 2017
Accepted 20 April 2017
Available online 28 April 2017

Keywords:
Inverse coefficient problem
Potential function
Inverse Sturm–Liouville operator
Energetic boundary function method
Iterative method

ABSTRACT

We consider an inverse problem for reconstructing the second-order Sturm–Liouville operator with the help of boundary data. The concept of energetic boundary functions, which satisfy the given boundary conditions and preserve the energy, is introduced for the first time. Then, we can derive a linear system to recover the unknown important coefficients of leading coefficient and potential function through a few iterations. Two examples are given to verify the iterative method.

1. Introduction

In this paper we are going to use a simple mathematical method to treat the following inverse coefficient problem. Find the pair of unknown functions $\langle y(x), \alpha(x)\rangle$ in the problem

$$-(\alpha(x)y'(x))' + q(x)y(x) = H(x), \quad x \in (0, \ell), \tag{1}$$

$$y(0) = y_0, \quad y(\ell) = y_\ell, \quad y'(0) = y'_0, \quad y'(\ell) = y'_\ell, \tag{2}$$

$$\alpha(0) = \alpha_0, \quad \alpha(\ell) = \alpha_\ell, \tag{3}$$

from the boundary measured data of y_0, y_ℓ, y'_0, y'_ℓ, α_0 and α_ℓ, where $q(x)$ and $H(x)$ are given functions. The Sturm–Liouville operator needs to be written in the following form $Ly := -(\alpha(x)y'(x))' + q(x)y(x)$, since in

* Corresponding author.
E-mail address: leedonlion408@163.com (B. Li).

http://dx.doi.org/10.1016/j.aml.2017.04.023

this form it satisfies the positivity condition $(Ly, y) \geq c_0\|y\|_1$, when $\alpha(x) \geq c_0 > 0$, for all $x \in (0, \ell)$. Here $\|\cdot\|_1$ is the norm of the Sobolev space $H^1(0, \ell)$.

Multiplying Eq. (1) by $y(x)$, integrating it from $x = 0$ to $x = \ell$, using integration by parts, and in view of Eqs. (2) and (3), we can obtain the following energy identity:

$$\int_0^\ell [\alpha(x)y'(x)^2 + q(x)y^2(x) - H(x)y(x)]dx = \alpha_\ell y_\ell y'_\ell - \alpha_0 y_0 y'_0 := d_0, \tag{4}$$

of which the deformation energy, potential energy and external work are balanced to a constant d_0. This equation is useful for the reconstruction of the coefficients $\alpha(x)$ and $q(x)$.

The inverse coefficient problem as specified above is a severely ill-posed problem, since similar boundary data may correspond to significantly different coefficients. The inverse problem related to the determination of the leading coefficient of the Sturm–Liouville equation from the boundary measurements has been studied by Hasanov and Shores [1]. An analysis of the problem (1)–(3) and also numerical methods have been developed by Kaltenbacher [2], Hasanov and Pektas [3], Seyidmamedov and Hasanov [4], Hasanov and Seyidmamedov [5], and Hasanov [6]. Furthermore, Cui et al. [7] considered the nonlinear Sturm–Liouville boundary value problem, and obtained a global bifurcation result for a related bifurcation problem. Liu [8] has developed a Lie-group adaptive method to identify $\alpha(x)$, which is resorted on the boundary data, and Liu [9] solved the one-dimensional Calderón problem by using the Lie-group adaptive method.

2. Energetic boundary function method

Usually, we cannot exactly know $y(x)$ in Eq. (1), because $\alpha(x)$ is an unknown function. However, we can build up some functions to approximate $y(x)$ to our best. First, we can derive the zero-th function:

$$B_0(x) = \frac{1}{\ell^3}[2y_0 - 2y_\ell + y'_0\ell + y'_\ell\ell]x^3 - \frac{1}{\ell^2}[3y_0 - 3y_\ell + 2y'_0\ell + y'_\ell\ell]x^2 + y'_0 x + y_0, \tag{5}$$

which automatically satisfies the boundary conditions in Eq. (2).

Then, we can further set up the jth homogeneous boundary function:

$$B_j(x) = (x^4 - 2\ell x^3 + \ell^2 x^2)x^{j-1}, \; j \geq 1. \tag{6}$$

They are at least the fourth-order boundary functions, satisfying the following homogeneous boundary conditions:

$$y(0) = 0, \quad y(\ell) = 0, \quad y'(0) = 0, \quad y'(\ell) = 0. \tag{7}$$

The set of

$$\{B_0(x), B_j(x)\}, \quad j \geq 1 \tag{8}$$

constitutes an affine linear space of boundary functions, allowing the combination of $B_0(x)$ and $B_j(x)$ to be an affine linear element:

$$E_j(x) = B_0(x) + \gamma_j B_j(x), \quad j \geq 1. \tag{9}$$

Now the problem is how to determine γ_j for each element of $E_j(x)$.

Because $E_j(x)$ already automatically satisfies the boundary conditions in Eq. (2), i.e.,

$$E_j(0) = y_0, \quad E_j(\ell) = y_\ell, \quad E'_j(0) = y'_0, \quad E'_j(\ell) = y'_\ell, \tag{10}$$