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Multiple solutions for the fourth-order elliptic equation with vanishing potential[☆]

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A B S T R A C T

This paper is concerned with the following fourth-order elliptic equation

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = K(x)f(u) + \mu\xi(x)|u|^{p-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $N \geq 5$, V, K are nonnegative continuous functions and f is a continuous function with a quasicritical growth. By working in weighted Sobolev spaces and using a variational method, we prove that the above equation has two nontrivial solutions.

1. Introduction

This paper is concerned with the following fourth-order elliptic equation

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = K(x)f(u) + \mu\xi(x)|u|^{p-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $N \geq 5$, $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $\mu > 0$, $1 < p < 2$, $V, K \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R})$.

The fourth-order elliptic problems have deep mathematical and physical background. Problem (1.1) arises in the study of traveling waves in suspension bridge and the study of the static deflection of an elastic plate in a fluid, see [1,2]. Over the past decades, problem (1.1) and problems similar as (1.1) have captured a lot of interest, many authors have shown their interest in elliptic equation and system both in bounded domains

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and unbounded domains, see [3–8]. In these literatures, we find that the authors have considered the existence and multiplicity of solutions for problem (1.1), and we observe that various interesting conditions on V have been studied. In the present paper, the condition on V we will consider is the zero mass case, which occurs with the potential V vanishing at infinity, that is

$$\lim_{|x|\to+\infty} V(x) = 0.$$

In [9], Ambrosetti, Felli and Malchiodi considered this case for Schrödinger equation when

$$f(s) = s^m, \text{ with } 2 < m < \frac{N+2}{N-2},$$

$V, K : \mathbb{R}^N \to \mathbb{R}$ are smooth functions and there exist constants $a_1, a_2, a_3, A, k_1 > 0$ such that

(VK) $\frac{a_3}{1+|x|^{a_1}} \leq V(x) \leq A$ and $0 < K(x) \leq \frac{k_1}{1+|x|^{a_2}}, \ \forall x \in \mathbb{R}^N$

and such that a_1, a_2 satisfy

$$\frac{N+2}{N-2} - \frac{4a_2}{a_1(N-2)} < p \text{ if } 0 < a_2 < a_1 \text{ or } m > 1, \text{ where } a_2 > a_1.$$

Later, in [10], Ambrosetti and Wang also considered the condition (VK), but the condition on V was assumed only outside of a ball centered at origin. In [11], Alves and Souto considered a more general condition on V and K, from which the working space can be embedded into the weighted space. Using the ideas in [11], Deng and Shuai obtained nontrivial solutions for a semilinear biharmonic problem with critical growth and potential vanishing at infinity in [12].

Motivated by the above papers, in the present paper, we consider biharmonic problem with mixed nonlinearity and the potential V vanishing at infinity. An interesting problem of this paper is a more general mixed nonlinearity involving a combination of superlinear terms $f(u)$ and sublinear terms $\xi(x)|u|^{p-2}u$ $(1 < p < 2)$. To the best of knowledge, few works concerning on this case up to now. Moreover, our results can be applied to the concave and convex nonlinear term case. To this end, we need some assumptions on V, K and f. Here, we say that $(V, K) \in \mathcal{K}$ if the following conditions hold:

(V) $V(x) > 0, \forall x \in \mathbb{R}^N$ and $\lim_{|x|\to+\infty} V(x) = 0$, (shortly $V(\infty) = 0$);

(K_1) (i) $K(x) > 0, \forall x \in \mathbb{R}^N$, and $K \in L^\infty(\mathbb{R}^N)$;

(ii) If $\{B_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|B_n| \leq R$ for all n and some $R > 0$, then

$$\lim_{r\to+\infty} \int_{B_n \cap B_r^c(0)} K(x)dx = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Moreover, one of the following conditions occurs:

(K_2) $\frac{K}{V} \in L^\infty(\mathbb{R}^N)$, or

(K_3) there is $\sigma \in (2, 2_*)$ such that

$$\frac{K(x)}{[V(x)]^{\frac{2_*-\sigma}{2_*-2}}} \to 0 \text{ as } |x| \to +\infty, \text{ where } 2_* = \frac{2N}{N-4} \text{ for } N \geq 5.$$

In order to state our results, we need the following assumptions for the term $f(u)$:

(F_1) $\limsup_{u\to0} \frac{f(u)}{u} = 0$ if (K_2) holds, or $\limsup_{u\to0^+} \frac{f(x,u)}{u^{\sigma-1}} < +\infty$ if (K_3) holds;

(F_2) f has a quasicritical growth, that is, $\limsup_{u\to+\infty} \frac{f(u)}{u^{2_*-1}} = 0$;

(F_3) there exists $\theta > 2$ such that $0 \leq \theta F(u) \leq uf(u)$ for all $u \in \mathbb{R}$, where $F(u) = \int_0^u f(s)ds$.