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Numerical approximation of the basic reproduction number for a class of age-structured epidemic models

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A B S T R A C T

We are concerned with the numerical approximation of the basic reproduction number $\mathcal{R}_0$, which is the well-known epidemiological threshold value defined by the spectral radius of the next generation operator. For a class of age-structured epidemic models in infinite-dimensional spaces, $\mathcal{R}_0$ has the abstract form and cannot be explicitly calculated in general. We discretize the linearized equation for the infective population into a system of ordinary differential equations in a finite n-dimensional space and obtain a corresponding threshold value $\mathcal{R}_{0,n}$, which can be explicitly calculated as the positive dominant eigenvalue of the next generation matrix. Under the compactness of the next generation operator, we show that $\mathcal{R}_{0,n} \to \mathcal{R}_0$ as $n \to +\infty$ in terms of the spectral approximation theory.

1. Introduction

In the field of mathematical epidemiology, the age-structure of population has been regarded important since most infectious diseases such as childhood diseases and sexually transmitted diseases have age-dependent characteristics. Age-structured SIR epidemic models, in which the population is divided into three subpopulations called susceptible, infective and recovered, are one of the most basic epidemic models and have attracted much attention of researchers for decades [1–8]. In [3], Greenhalgh conjectured that the spectral radius of a certain linear integral operator would play the role of a threshold value for the existence and stability of each equilibrium in an age-structured SIR epidemic model. In [5], Inaba proved his conjecture: if the threshold value is less than one, then the disease-free equilibrium is globally asymptotically stable and there exists no endemic equilibrium, whereas if it is greater than one, then the disease-free equilibrium is unstable and the endemic equilibrium uniquely exists and it is locally asymptotically stable under some additional conditions. The threshold value is nowadays called the basic reproduction number $\mathcal{R}_0$, and its epidemiological meaning is the expected value of secondary cases produced by a typical infective individual during its entire infectious period in a fully susceptible population [9].

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$\mathcal{R}_0$ is not only mathematically but also epidemiologically important for assessing the disease burden of infectious diseases. However, for age-structured epidemic models in infinite-dimensional spaces, $\mathcal{R}_0$ cannot be explicitly calculated in general since it has the abstract form as the spectral radius of the linear integral operator called the next generation operator. If we discretize the model into a finite dimensional space, then the corresponding threshold value can be explicitly calculated as the positive dominant eigenvalue of the nonnegative irreducible matrix called the next generation matrix. We can expect that the corresponding threshold value converges to $\mathcal{R}_0$ as the step size of the discretization decreases. However, the convergence of the eigenvalues with preservation of the algebraic multiplicity is not trivial and we need the spectral approximation theory [10] to show it mathematically rigorously. In this study, in terms of the spectral approximation theory, we show that the corresponding threshold value $\mathcal{R}_{0,n}$ in the n-dimensional space converges to $\mathcal{R}_0$ as $n \to +\infty$, provided the next generation operator is compact. The compactness holds under relatively weak conditions. By using the demographic data in Japan, we give a numerical example to illustrate the theoretical result. For other studies of age-structured epidemic models from the numerical viewpoints, see [11–14]. Although these studies focused on the convergence of numerical solutions, our focus in this paper is on the convergence of the spectral approximation for the computation of $\mathcal{R}_0$. For another study of the spectral approximation for age-structured population models, see [15]. For another study of the approximation of $\mathcal{R}_0$ for epidemic models in time periodic environment, see [16].

2. Main result

We consider the following equation for the infective population, which is linearized around the disease-free steady state.

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) I(t,a) = S^0(a)\displaystyle\int_0^{a_\dagger} \beta(a,\sigma) I(t,\sigma)d\sigma - (\mu(a)+\gamma(a))I(t,a), \quad t>0,\ a\in(0,a_\dagger), \\ I(t,0)=0, \quad t>0, \qquad I(0,a)=I_0(a), \quad a\in(0,a_\dagger), \end{cases} \tag{2.1}$$

where $I(t,a)$ denotes the infective population of age a at time t, $S^0(a)$ denotes the susceptible population of age a in the disease-free steady state, $a_\dagger > 0$ denotes the maximum age, $\beta(a,\sigma)$ denotes the disease-transmission coefficient, $\mu(a)$ denotes the force of mortality, $\gamma(a)$ denotes the recovery rate and $I_0(a)$ denotes the initial age distribution of the infective population. Note that the vertical transmission is excluded since $I(t,0) = 0$ for all $t > 0$. We assume that S^0, β, μ and γ are continuous, strictly positive and uniformly bounded on $[0, a_\dagger]$. We define the following two linear operators on $X := L^1(0, a_\dagger)$.

$$\begin{cases} A\varphi(a) := -\dfrac{d}{da}\varphi(a) - (\mu(a)+\gamma(a))\,\varphi(a), \quad D(A) := \Big\{\varphi \in X:\ \varphi \text{ is absolutely continuous on } [0,a_\dagger], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dfrac{d}{da}\varphi \in X \text{ and } \varphi(0)=0\Big\}, \\ F\varphi(a) := S^0(a)\displaystyle\int_0^{a_\dagger}\beta(a,\sigma)\varphi(\sigma)d\sigma. \end{cases}$$

Using A and F, we can rewrite (2.1) into the following abstract Cauchy problem in X.

$$\frac{d}{dt}I(t) = AI(t) + FI(t), \quad I(0) = I_0 \in D(A). \tag{2.2}$$

It is easy to see that the positive inverse $(-A)^{-1}$ is defined as $(-A)^{-1}\varphi := \int_0^a e^{-\int_\sigma^a(\mu(\eta)+\gamma(\eta))d\eta}\varphi(\sigma)d\sigma$, $\varphi \in X$ and the next generation operator K is defined as follows (see, for instance, [5]).

$$K\varphi(a) := F(-A)^{-1}\varphi(a) = S^0(a)\int_0^{a_\dagger}\beta(a,\sigma)\int_0^\sigma e^{-\int_\rho^\sigma(\mu(\eta)+\gamma(\eta))d\eta}\varphi(\rho)d\rho\, d\sigma, \quad \varphi \in X.$$