Applied Mathematics Letters 73 (2017) 113–119

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Discontinuous Galerkin methods for the incompressible flow with nonlinear leak boundary conditions of friction type[☆]

Feifei Jing[a], Jian Li[b,c], Zhangxin Chen[d,e,*], Wenjing Yan[d]

[a] *School of Sciences, Xi'an University of Technology, Xi'an, 710054, China*
[b] *Department of Mathematics, School of Arts and Sciences, Shaanxi University of Science and Technology, Xi'an 710021, China*
[c] *Institute of Computational Mathematics and its Applications, Baoji University of Arts and Sciences, Baoji 721007, China*
[d] *School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China*
[e] *Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N. W. Calgary, Alberta, T2N 1N4, Canada*

ARTICLE INFO

Article history:
Received 24 February 2017
Received in revised form 26 March 2017
Accepted 26 March 2017
Available online 27 April 2017

Keywords:
Stokes equations
Variational inequality
DG methods
Error estimates

ABSTRACT

This work aims at employing the discontinuous Galerkin (DG) methods for the incompressible flow with nonlinear leak boundary conditions of friction type, whose continuous variational problem is an inequality due to the subdifferential property of such boundary conditions. Error estimates are derived for the velocity and pressure in the corresponding norms, this work ends with numerical results to demonstrate the theoretically predicted convergence orders and reflect the leak or non-leak phenomena.

1. Introduction

In this paper, we are interested in the model of the incompressible flow governed by the Stokes equations with nonlinear leak boundary conditions of friction type (LBCF) [1–3]. We firstly introduce some symbols and notation. Let $\Omega \subset R^d$ be an open bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$. We assume $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_S$, $\Gamma_D \cap \Gamma_S = \emptyset$ with both Γ_D and Γ_S non-empty. Let $\mathbf{n} = (\mathrm{n}_1, \mathrm{n}_2)^T$ be the unit outward normal on Γ_S, and $\boldsymbol{\tau}$ be the unit tangent vector, then we write $\mathrm{v}_{\mathbf{n}} = \mathbf{v} \cdot \mathbf{n}, \mathbf{v}_{\boldsymbol{\tau}} = \mathbf{v} - \mathrm{v}_{\mathbf{n}}\mathbf{n}$ for the normal and tangential components of a vector $\mathbf{v}$ defined on the boundary. $\mathbf{u}$ and p denote the velocity of fluid and pressure, respectively, $\nu > 0$ is the viscosity constant, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})(1 \le i, j \le d)$ denotes the

☆ This work is supported by the National Natural Science Fund of China (Nos. 11371031, 11371288) and is partially supported by Foundation CMG in Xi'an Jiaotong University.

* Corresponding author at: Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N. W. Calgary, Alberta, T2N 1N4, Canada.
E-mail address: zhachen@ucalgary.ca (Z. Chen).

http://dx.doi.org/10.1016/j.aml.2017.03.017
0893-9659/

strain tensor, and $\mathbb{T}_{ij}(\mathbf{u}, p) = 2\nu\varepsilon_{ij}(\mathbf{u}) - p\delta_{ij}$ is the stress tensor, where $\delta_{ij} = 1$, if $i = j$, otherwise $\delta_{ij} = 0$. The two-dimensional stationary Stokes equations are

$$-\nabla \cdot \mathbb{T}_{ij}(\mathbf{u}, p) = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\mathbf{f}$ is the given external force. Throughout this paper, the boldface symbols denote vector-valued quantities. Over Γ_D, we specify the homogeneous Dirichlet boundary condition (b.c.):

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D. \tag{1.2}$$

We define the stress vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^T$ by $\sigma_i = \sigma_i(\mathbf{u}, p) = \sum_{j=1}^{2} \mathbb{T}_{ij}(\mathbf{u}, p)\mathrm{n}_j$. Then $\sigma_{\mathbf{n}} = \boldsymbol{\sigma} \cdot \mathbf{n}$ for its normal component. Thus over Γ_S, we specify a leak b.c.s of friction type:

$$\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{0}, \quad |\sigma_{\mathbf{n}}| \le g, \quad \sigma_{\mathbf{n}}\mathrm{u}_{\mathbf{n}} + g|\mathrm{u}_{\mathbf{n}}| = 0 \quad \text{on } \Gamma_S. \tag{1.3}$$

The positive function $g : \Gamma_S \to [0, \infty)$ is known as the friction function. This type of b.c.s was first introduced in [1] for applications in the oil flow over or beneath sand layers.

Since the frictional b.c.s are treated mainly in the context of elasticity [4], and in the pioneer work [1,2], the author proposed the nonlinear slip b.c.s of friction type (SBCF) and LBCF in fluid dynamics, where the SBCF can be obtained by replacing $\mathbf{n}$ with $\boldsymbol{\tau}$ and vice versa in (1.3), which has drawn many experts' attention in the past two decades [2,5–9,18,19]. Whereas, LBCF, in which case a material is allowed to penetrate the boundary but not allowed unless the flow is strong enough, is also worth considering in the context of fluid dynamics. DG methods [10–12], which relax the continuity of approximation functions across the finite element boundaries, are allowed to be easily implemented on highly unstructured meshes and also possess locality and flexibility. DG methods have been used to solve variational inequality problems [5,13], however, to our knowledge, there has been rare analysis of DG methods for the Stokes equations with LBCF. In this paper, following the unified framework developed in [10,13], we present the interior penalty DG methods for the problem (1.1)–(1.3), and explore the error estimates for the proposed methods.

An outline of this work is as follows. In Section 2, the continuous variational form, the DG schemes and existence and uniqueness of discrete solutions are exhibited. Error estimates are derived in Section 3. This paper ends with a section on numerical results to illustrate the theoretical results.

2. Variational inequality problem

In this section, we give the continuous variational forms of Eqs. (1.1)–(1.3) with LBCF, propose the DG scheme to the continuous problem, and recall some results of the discrete bilinear forms.

2.1. Continuous variational inequality problem

Before explaining what our DG discrete scheme aims at, we review the results shown in the study of a weak form for (1.1)–(1.3) as the following variational inequality [2]: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{cases} a(\mathbf{u}, \boldsymbol{v} - \boldsymbol{u}) \; - d(\boldsymbol{v} - \boldsymbol{u}, p) + j(\mathrm{v}_{\mathbf{n}}) - j(\mathrm{u}_{\mathbf{n}}) \ge (\mathbf{f}, \boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad d(\mathbf{u}, q) = 0 \quad \forall\, q \in Q. \end{cases} \tag{2.1}$$

Here, the standard notation of Sobolev spaces [14] are employed, we define the proper function spaces, bilinear terms and barrier term which appear above as:

$$\mathbf{V} = \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v}|_{\Gamma_D} = \mathbf{0}, \mathbf{v}_{\boldsymbol{\tau}}|_{\Gamma_S} = \mathbf{0}\},\ \mathbf{Y} = \boldsymbol{L}^2(\Omega),\ Q \triangleq L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \,\mathrm{d}\mathbf{x} = 0\},$$

$$a(\mathbf{u}, \mathbf{v}) = 2\nu \sum_{i,j=1}^{2} \int_\Omega \varepsilon_{ij}(\mathbf{u}) : \varepsilon_{ij}(\mathbf{v}) \,\mathrm{d}\mathbf{x},\ d(\mathbf{v}, p) = \int_\Omega \operatorname{div} \mathbf{v}\, q \,\mathrm{d}\mathbf{x},\ j(\mathrm{v}_{\mathbf{n}}) = \int_{\Gamma_S} g|\mathrm{v}_{\mathbf{n}}| \,\mathrm{d}s\ \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}, p \in Q,$$