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A note on global stability for a tuberculosis model[☆]

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ABSTRACT

In this note, we investigate the global stability for a tuberculosis model with vaccination and treatment. The model exhibits the traditional threshold behavior. We prove that the basic reproduction number characterizes the global dynamics of the model, that is, the disease endemic equilibrium of the model is globally asymptotically stable whenever it exists, which accords with the corresponding conjecture presented in Liu and Zhang (2011).

1. Introduction

To predict the potential public health impact of new tuberculosis (TB) vaccine and treatment for epidemic control in high-incidence countries, a new deterministic model with vaccination and treatment was proposed and analyzed mathematically in Liu and Zhang [1] as follows:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta S(I + \rho_1 T) - (\mu + p)S, \\ \dfrac{dV}{dt} = pS - \rho_2 \beta V(I + \rho_1 T) - \mu V, \\ \dfrac{dL}{dt} = l\beta S(I + \rho_1 T) + \rho_2 \beta V(I + \rho_1 T) - (\mu + \delta)L + \rho T, \\ \dfrac{dI}{dt} = (1 - l)\beta S(I + \rho_1 T) + \delta L - (\mu + \alpha + \gamma)I, \\ \dfrac{dT}{dt} = \gamma I - (\mu + \rho)T, \\ N(t) = S(t) + V(t) + L(t) + I(t) + T(t), \end{cases} \tag{1.1}$$

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where the total population at time t, denoted by $N(t)$, is partitioned into subclasses of individuals who are susceptible ($S(t)$), vaccinated ($V(t)$), infected with TB in latent (asymptomatic) stage ($L(t)$), infected with TB in the active stage ($I(t)$) and treated individuals infected with TB ($T(t)$). The parameter Λ represents the constant recruitment rate of susceptible individuals and β is the disease transmission coefficient. The parameter $\rho_1 < 1$ accounts for the reduction in infectiousness among individuals with active TB who are treated, μ is the natural death rate and p is the vaccination rate. The parameter $\rho_2 < 1$ represents the reduction in risk of infection due to vaccination and l is the fraction of susceptible individuals who acquire TB infection moves to the latent TB class (L). The parameter δ is the rate at which an individual leaves the latent compartment to become infectious and ρ is the rate of individuals successfully treated of TB return to the latent TB stage. The parameter α is the disease-induced death rate coefficient for individuals in compartment I and γ is the rate at which an infective individual leaves the infectious class and transferred to the treatment class.

Let

$$D = \left\{ (S, V, L, I, T) \in [0, +\infty)^5 : S \le \frac{\Lambda}{\mu + p}, V \le \frac{\Lambda p}{\mu(\mu + p)}, N \le \frac{\Lambda}{\mu} \right\}.$$

Liu and Zhang [1] proved that D is positive-invariant and attracting, and the basic reproduction number for model (1.1) can be given by

$$R_0 = \frac{(\mu + \rho + \gamma\rho_1)(\beta\delta\rho_2 V^0 + \beta\delta S^0 + \beta\mu(1 - l)S^0)}{(\mu + \delta)(\mu + \rho)(\mu + \alpha + \gamma) - \delta\rho\gamma}$$

with $S^0 = \frac{\Lambda}{\mu+p}$ and $V^0 = \frac{\Lambda p}{\mu(\mu+p)}$. They also showed that model (1.1) has only a disease-free equilibrium $E_0 = (S^0, V^0, 0, 0, 0)$ and for $R_0 > 1$, there exists a unique endemic equilibrium $E^* = (S^*, V^*, L^*, I^*, T^*)$.

Furthermore, Liu and Zhang [1] considered the global stability of endemic equilibrium only for a special case of model (1.1): $\rho_2 = 0$, i.e. the vaccine efficacy is 100%. More precisely, they studied the following system

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta S(I + \rho_1 T) - (\mu + p)S, \\ \dfrac{dV}{dt} = pS - \mu V, \\ \dfrac{dL}{dt} = l\beta S(I + \rho_1 T) - (\mu + \delta)L + \rho T, \\ \dfrac{dI}{dt} = (1 - l)\beta S(I + \rho_1 T) + \delta L - (\mu + \alpha + \gamma)I, \\ \dfrac{dT}{dt} = \gamma I - (\mu + \rho)T \end{cases} \tag{1.2}$$

and proved the following result.

Theorem 1.1 (*Theorem 4.2 of [1]*). *If the basic reproduction number*

$$\tilde{R}_0 = \frac{(\mu + \rho + \gamma\rho_1)(\beta\delta + \beta\mu(1 - l))\Lambda}{(\mu + p)[(\mu + \delta)(\mu + \rho)(\mu + \alpha) + \mu\gamma(\mu + \rho + \delta)]}$$

of system (1.2) satisfies $\tilde{R}_0 > 1$, then system (1.2) has a unique endemic equilibrium $E^* = (S^*, V^*, L^*, I^*, T^*)$ which is globally asymptotically stable on $(0, +\infty)^5 \setminus \mathscr{A}$, with solutions in $\mathscr{A}$ limiting to E_0, where $\mathscr{A} = \{(S, V, 0, 0, 0)\}$ and

$$S^* = \frac{\Lambda}{\mu + p + \beta\frac{\mu+\rho+\gamma\rho_1}{\mu+\rho} I^*}, \quad V^* = \frac{pS^*}{\mu}, \quad L^* = \frac{\beta l S^*(I^* + \rho_1 T^*) + \rho T^*}{\mu + \delta},$$
$$I^* = \frac{(\mu + p)(\mu + \rho)}{\beta(\mu + \rho + \rho_1\gamma)}(R_0 - 1), \quad T^* = \frac{\gamma}{\mu + \rho} I^*.$$

At the end of Section 4 of [1], Liu and Zhang pointed out that "For the full model, we guess that the endemic equilibrium E^* is also globally stable as $R_0 > 1$. To prove this result, we may need new Lyapunov