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# Existence and uniqueness of the modified error function

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A B S T R A C T

This article is devoted to the proof of the existence and uniqueness of the modified error function introduced in Cho and Sunderland (1974). This function is defined as the solution to a nonlinear second order differential problem depending on a real parameter. We prove here that this problem has a unique non-negative analytic solution when the parameter assumes small positive values.


## 1. Introduction

In 1974, Cho and Sunderland [1] studied a solidification process with temperature-dependent thermal conductivity and obtained an explicit similarity solution in terms of what they called a *modified error function*. This function is defined as the solution to the following nonlinear differential problem:

$$[(1+\delta y(x))y'(x)]' + 2xy'(x) = 0 \quad 0 < x < +\infty \tag{1a}$$

$$y(0) = 0 \tag{1b}$$

$$y(+\infty) = 1 \tag{1c}$$

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where $\delta \geq -1$ is given. Graphics for numerical solutions of (1) for different values of $\delta$ can be found in [1]. The classical error function is defined by:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)dz, \quad x > 0, \tag{2}$$

and it is a solution to (1) when $\delta = 0$. This makes meaningful the denomination *modified error function* given for the solution to problem (1).

The modified error function has also appeared in the context of diffusion problems before 1974 [2,3]. It was also used later in several opportunities to find similarity solutions to phase-change processes, e.g. see [4,5]. It was cited in [6], where several nonlinear ordinary differential problems arise from a wide variety of fields are presented. Closed analytical solutions for Stefan problems with variable diffusivity is given in [7]. Temperature-dependent thermal coefficients are very important in thermal analysis, e.g. see [8]. Nevertheless, to the knowledge of the authors, the existence and uniqueness of the solution to problem (1) has not been yet proved. In general, the existence theorems for boundary value problems for second order ordinary differential equations include certain continuity and bounded derivatives that are not guaranteed for problem (1), even when it is reduced to a bounded domain (see, for example, [9–12]). This article is devoted to prove it for small $\delta > 0$ using a fixed point strategy.

## 2. Existence and uniqueness of solution to problem (1)

The main idea developed in this Section is to study problem (1) through the linear problem given by the differential equation:

$$[(1 + \delta \Psi_h(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \tag{1a$^\star$}$$

and conditions (1b), (1c). The function $\Psi_h$ in (1a$^\star$) is defined by:

$$\Psi_h(x) = 1 + \delta h(x), \quad x > 0, \tag{3}$$

where $\delta > 0$, $h \in K \subset X$ is given and:

$$X = \left\{h : \mathbb{R}_0^+ \to \mathbb{R} \,/\, h \text{ is an analytic function}, \, \|h\|_\infty < \infty\right\} \tag{4a}$$

$$K = \{h \in X \,/\, \|h\|_\infty \leq 1, \, 0 \leq h, \, h(0) = 0, \, h(+\infty) = 1\}. \tag{4b}$$

Hereinafter, we will refer to the problem given by (1a$^\star$), (1b) and (1c) as problem (1$^\star$). Let us observe that $K$ is non-empty closed subset of the Banach space $X$.

The advantage in considering the linear equation (1a$^\star$) is that it can be easily solved through the substitution $v = y'$. Thus, we have the following result:

**Theorem 2.1.** *Let $h \in K$ and $\delta > 0$. The solution $y$ to problem (1$^\star$) is given by:*

$$y(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_h(\xi)} d\xi\right) d\eta \quad x \geq 0, \tag{5}$$

*where the constant $C_h$ is defined by:*

$$C_h = \left(\int_0^{+\infty} \frac{1}{\Psi_h(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_h(\xi)} d\xi\right) d\eta\right)^{-1}. \tag{6}$$