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Regularity of Navier–Stokes flows with bounds for the pressure

Chuong V. Tran[a,*], Xinwei Yu[b]

[a] School of Mathematics and Statistics, University of St Andrews, St Andrews KY16 9SS, United Kingdom
[b] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada

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ABSTRACT

This study derives regularity criteria for solutions of the Navier–Stokes equations. Let $\Omega(t) := \{x : |u(x,t)| > c\,\|u\|_{L^r(\mathbb{R}^3)}\}$, for some $r \geq 3$ and constant c independent of t, with measure $|\Omega|$. It is shown that if $\|p + \mathcal{P}\|_{L^{3/2}(\Omega)}$ becomes sufficiently small as $|\Omega|$ decreases, then $\|u\|_{L^{(r+6)/3}(\mathbb{R}^3)}$ decays and regularity is secured. Here p is the physical pressure and $\mathcal{P}$ is a pressure moderator of relatively broad forms. The implications of the results are discussed and regularity criteria in terms of bounds for $|p + \mathcal{P}|$ within Ω are deduced.

1. Introduction

This note is concerned with the Cauchy problem of the Navier–Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \Delta u, \qquad \nabla \cdot u = 0 \tag{1}$$

in $\mathbb{R}^3 \times (0, \infty)$ with $u(x, 0) = u_0(x)$ divergence-free, smooth and decaying sufficiently fast at infinity. Given such an initial velocity field, it is well known that a classical solution exists up to some finite time $t = T$, which depends on $u_0(x)$. The question is whether or not the solution remains smooth (regular) beyond T, particularly up to all $t \geq T$ (global regularity).

Decades of active research on this problem since Leray's seminal work in the 1930s have resulted in a rich literature [1–33]. Yet, the prospect of a definitive answer to the above question has become increasingly remote. Early studies by Prodi [22], Serrin [24] and Ladyzhenskaya [20] found that regularity is guaranteed provided that $\int_0^T \|u\|_{L^r}^{2r/(r-3)}\,\mathrm{d}t < \infty$, for $r \in (3, \infty)$. Recently, Escauriaza, Seregin and Sverák [14] have extended this criterion to $\operatorname{esssup}_{t\in(0,T)} \|u\|_{L^3} < \infty$, for the critical case $r = 3$. Various criteria expressible in terms of the pressure p and its gradient ∇p have been derived by a number of authors

* Corresponding author.
E-mail addresses: cvt1@st-andrews.ac.uk (C.V. Tran), xinweiyu@math.ualberta.ca (X. Yu).

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[1,4,8,10,12,15,23,25,28,30,32]. Among these, two criteria are most relevant to the present context. One is the criterion

$$\int_0^T \|p\|_{L^r}^{2r/(2r-3)}\,\mathrm{d}\tau < \infty, \quad \text{for} \quad r \in (3/2, \infty), \tag{2}$$

which is an analogue of the Prodi–Serrin–Ladyzhanskaya result. The other is the following theorem by Seregin and Sverák [23].

Theorem 1 (*Seregin & Sverák, 2002*)**.** *Let u and p solve the Navier–Stokes equations (1). Let $g(x,t)$: $\mathbb{R}^3 \times (0,\infty) \to [0,\infty)$ be such that for any $t_0 > 0$, there exists $r = r(t_0) > 0$ such that*

$$\sup_{x_0\in\mathbb{R}^3}\ \sup_{t_0-r^2\le t\le t_0} \int_{B(x_0,r)} \frac{g(x,t)}{|x-x_0|}\,\mathrm{d}x < \infty.$$

If

$$\frac{|u(x,t)|^2}{2} + p(x,t) \le g(x,t) \tag{3}$$

or

$$p(x,t) \ge -2g(x,t), \tag{4}$$

for $x \in \mathbb{R}^3$ and $t \in (0,\infty)$, then $u(x,t)$ is smooth and unique.

This note derives regularity criteria in terms of bounds for an "effective" pressure. The results have some bearing on criteria (2) and (3) discussed in the preceding paragraph. Qualitatively speaking, it is proved that if an effective pressure in region(s) of high velocity is bounded by a singular function similar to $g(x,t)$, then regularity is secured. Here the effective pressure is the sum of the physical pressure and moderators, which may potentially be used to moderate the former in such region(s).

For the rest of this study, c denotes a positive constant, which may assume different values from one expression to another.

2. Results

Let $r \ge 3$ and $q := (r+6)/3$, so that $3 \le q \le r$. The evolution of $\|u\|_{L^q} := \|u\|_{L^q(\mathbb{R}^3)}$ is governed by

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^q}^q &= (q-2)\int_{\mathbb{R}^3} p|u|^{q-2}\,\widehat{u}\cdot\nabla|u|\,\mathrm{d}x \\ &\quad -(q-2)\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 - \left\||u|^{(q-2)/2}\nabla u\right\|_{L^2}^2,\end{aligned} \tag{5}$$

where $\widehat{u}$ is the unit vector in the direction of u. Our aim is to derive conditions under which the driving term in (5) becomes smaller than the corresponding dissipation terms, thereby implying a decay of $\|u\|_{L^q}$ and regularity. The following lemmas constitute an integral part of our derivation. These results are taken from Ref. [27] (see also Ref. [26]) with minor modifications.

Lemma 1. *Let*

$$\mathcal{P}(x,|u|,t) := \sum_{i=1}^n f_i(x,t)g_i(|u|,t), \tag{6}$$

where $u\cdot\nabla f_i(x,t) = 0$ and $g_i(\xi,t) \in C^1$, then

$$\int_{\mathbb{R}^3} \mathcal{P}\,|u|^{q-2}\,\widehat{u}\cdot\nabla|u|\,\mathrm{d}x = 0. \tag{7}$$