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Critical singular exponent and asymptotic estimates in the parabolic equations

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ABSTRACT

This paper considers both quenching and blowup phenomena to the coupled parabolic equations with zero Dirichlet boundary. Here, one component of the solution represents the density of some chemical and the other denotes its temperature in some ignition process. All the singular phenomena have been obtained including simultaneous and nonsimultaneous blow-up or quenching, which are classified completely by the exponents. The results extend the ones in the previous paper (Liu and Chan, 2011).

1. Introduction

In this paper, we study the coupled parabolic problem as follows,

$$\begin{cases} u_t = \Delta u + v^p(1-u)^{-m}, \quad v_t = \Delta v + v^n(1-u)^{-q}, & (x,t) \in B_R\times(0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial B_R\times(0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in B_R, \end{cases} \tag{1.1}$$

which is used to describe some thermal ignition process, where u represents the density of the chemical while v denotes its temperature (see [1,2] and the papers therein). For uniqueness, we assume that $p,\ n \geq 1,\ q,\ m > 1$. The initial data $u_0(x),\ v_0(x) \in C^{2+\alpha}(B_R)$, $\alpha \in (0,1)$, satisfy the compatibility conditions on ∂B_R with $B_R := \{x \in R^N : |x| < R\}$. Liu and Chan in [3] studied the simpler problem of (1.1) as follows,

$$u_t = \Delta u + \gamma v^p, \quad v_t = \Delta v + \mu(1-u)^{-q}, \quad (x,t) \in \Omega\times(0,T), \tag{1.2}$$

where $\gamma,\ \mu > 0,\ p \geq 1,\ q > 1$. They assumed the initial data guarantee that u and v are non-decreasing in time variable. Let λ_1 and ϕ be the first eigenvalue and the first eigenfunction respectively of $(-\Delta)$ with zero

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Dirichlet boundary satisfying $\|\phi\|_{L^1(\Omega)} = 1$. The results were obtained: (a) If $p = 1$ and $\min\{\gamma, \mu q\} > \lambda_1$, then v blows up and u quenches at the same time. (b) If $2 > p > 1$ and $E(0) \geq E^*$ where E^* is the first positive root of $\gamma 2^{-p} E^p - \lambda_1 E - \frac{(2-p)\mu q(q+1)}{4}\left(\frac{p}{2}\right)^{p/(2-p)} = 0$, then v blows up and u quenches simultaneously. (c) If $p = 2$ and $\gamma E(0)/2 > \lambda_1$, then v blows up and u quenches simultaneously. (d) If $p > 2$ and $E(0) \geq (\lambda_1 + \sqrt{\lambda_1^2 + \gamma\beta})2/\gamma$ with $\beta = \gamma(2/p)^{2/(p-2)}(p-2)/p$, then v blows up and u quenches simultaneously. (e) The *singular rates* are taken of the form,

$$c(T-t)^{(p+1)/(pq+1)} \leq 1 - \|u(\cdot,t)\|_\infty \leq C(T-t)^{(p+1)/(pq+1)}, \tag{1.3}$$

$$c(T-t)^{-(q-1)/(pq+1)} \leq \|v(\cdot,t)\|_\infty \leq C(T-t)^{-(q+1+2pq)/(pq+1)}. \tag{1.4}$$

The blowup versus quenching phenomena also have been founded by Deng and Zhao in [4,5] to

$$\begin{cases} u_t = \Delta u + u^p, & (x,t) \in \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \eta} = -u^{-q}, & (x,t) \in \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), & x \in \Omega; \end{cases} \qquad \begin{cases} u_t = \Delta u - u^{-q}, & (x,t) \in \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \eta} = u^p, & (x,t) \in \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

They obtained some sufficient conditions for the existence of quenching or blowup solutions. Moreover, the quenching or blowup rate is considered. For the other results about the Neumann problems, the readers can refer to [6–10] and the papers therein.

The readers can check that, as for (1.2), quenching and blowup must occur simultaneously, which means the density reaches 1 and the temperature tends to infinity at the same time. But, for (1.1), the coupled relationship between the two components might lead to more complicated singular phenomena. For example, the density of the chemical does not reach 1, but the temperature tends to infinity in finite time. To our knowledge, it has not been discussed before and the methods used in [3] were too hard to give clearer classification of the singularity of the solutions.

In this paper, we use the different methods to discuss the complete and optimal classification of the singularity in (1.1) and also study their asymptotic estimates. In the next section, we give the main results of the paper. At Section 3, the critical singular exponent (Theorem 2.1) is proved. At the last section, we show the proofs of Theorem 2.2 on blow-up or quenching rates.

2. Main results

By the similar procedure of (1.2), one can obtain system (1.1) owes a unique local classical solution (u, v) satisfying $u_0 \leq u \leq F$, $v_0 \leq v \leq I$, where F and I are the solutions of the ordinary problem

$$\begin{cases} F'(t) = I^p(1-F)^{-m}, \quad I'(t) = I^n(1-F)^{-q}, \quad t > 0, \\ F(0) = \max\{u_0(x), x \in \bar{B}_R\}, \quad I(0) = \max\{v_0(x), x \in \bar{B}_R\}. \end{cases}$$

It follows from Theorem 8.9.2 of [1] that the solutions either exist globally or only exist in finite time. Let the initial data be radially decreasing and satisfy, for some $\varepsilon > 0$,

$$\Delta u_0(x) + (1-\varepsilon)v_0^p(x)(1-u_0(x))^{-m}, \ \Delta v_0(x) + (1-\varepsilon)v_0^n(x)(1-u_0(x))^{-q} \geq 0, \quad x \in B_R. \tag{2.1}$$

The first result is about the *critical singular exponent* for the singular solutions.

Theorem 2.1.

(i) *There exist initial data such that u quenches and v remains bounded if and only if $m + 1 > q$.*
(ii) *There exist initial data such that u does not quench and v blows up if and only if $n > p + 1$.*
(iii) *Any singularity is u quenching with bounded v if and only if $m + 1 > q$ and $n \leq p + 1$.*