Applied Mathematics Letters 65 (2017) 70–76

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On the variation of the spectrum of a Hermitian matrix[☆]

Wen Li[a,*], Seak-Weng Vong[b]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, PR China
[b] Department of Mathematics, University of Macau, Macau, PR China

ARTICLE INFO

Article history:
Received 22 August 2016
Received in revised form 9 October 2016
Accepted 9 October 2016
Available online 14 October 2016

Keywords:
Hermitian matrix
Eigenvalue
Perturbation

ABSTRACT

In this paper, we consider the eigenvalue variation for any perturbation of Hermitian matrices, and we obtain two perturbation bounds. The first bound always improves the existing bound, and the second bound also improves the existing one under a suitable condition. A simple example is given for comparing these bounds.

1. Introduction

Numerical analysis in eigenvalue problems of matrices has been being a hot topic in matrix computations because of its applications such as engineering computations et al. Perturbation analysis for eigenvalues is of significance since it needs to study the sensitivity for solving eigenvalue problems. The classical bounds for the eigenpair perturbation are the Hoffman–Wielandt theorem for eigenvalues (see [1]) and the sin Θ theorem for eigenspaces (see [2]), respectively. Recently, some perturbation bounds for eigenvalues have still been studied; see e.g. [3–10]. In particular, some authors obtained the eigenvalue perturbation bounds for the Hermitian matrices (e.g., see [11,12]). Here we consider a variation of eigenvalues for a Hermitian matrix. First we introduce some notations.

Let $\mathcal{C}^{m\times n}$ be the set of $m\times n$ complex matrices. By A^* we denote the conjugate transpose of a matrix A. The Frobenius norm of a matrices $\cdot$ is denoted by $\|\cdot\|_F$. The spectrum of A is denoted by $\{\lambda_1,\ldots,\lambda_n\}$.

Let $\widetilde{A}$ be a matrix with Schur's decomposition:

$$\widetilde{A}=\widetilde{U}(\widetilde{\Lambda}+\Delta)\widetilde{U}^*, \tag{1.1}$$

where $\widetilde{\Lambda}=diag(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_n)$ and Δ is strictly upper triangular, and $\widetilde{U}$ is unitary.

☆ The work was supported in part by National Natural Science Foundation of China (Nos. 11271144, 11671158) and by University of Macau (Grant No. MYRG2015-00064-FST).

* Corresponding author.
E-mail address: liwen@scnu.edu.cn (W. Li).

http://dx.doi.org/10.1016/j.aml.2016.10.005
0893-9659/

The eigenvalue perturbation bound for the normal matrices is the Hoffman–Wielandt theorem (see [1]), i.e., when both A and $\widetilde{A}$ are normal matrices, then there exists a permutation τ in $\{1,\dots,n\}$ such that

$$\left(\sum_{i=1}^{n}|\widetilde{\lambda}_{\tau(i)}-\lambda_i|^2\right)^{\frac{1}{2}}\le\|E\|_F. \tag{1.2}$$

If A is Hermitian, but its perturbed matrix $\widetilde{A}$ is arbitrary. The perturbation bound for eigenvalues is given below (e.g., see the bound (4.42) in Chapter 4 of [13] or [14]): there exists a permutation τ in $\{1,\dots,n\}$ such that

$$\left(\sum_{i=1}^{n}|\widetilde{\lambda}_{\tau(i)}-\lambda_i|^2\right)^{\frac{1}{2}}\le\sqrt{2}\|E\|_F, \tag{1.3}$$

and $\sqrt{2}$ is the best factor in (1.3). However, the bound (1.3) is independent of the quantity $\|\widetilde{A}-\widetilde{A}^*\|_F$, i.e., no matter how $\widetilde{A}^*$ closes to $\widetilde{A}$, this bound is the same. By this motivation, here we give some improvements for the bound (1.3). The new bound can reduce to the one in (1.2) when the perturbation is still Hermitian, also the new bound is sharper than the one in (1.3).

2. The bounds

Let M be an $n\times n$ matrix. By $\mathcal{D}(M)$, $\mathcal{L}(M)$ and $\mathcal{U}(M)$ we denote the diagonal part, strictly lower triangular part and strictly upper triangular part of M, i.e., $M=\mathcal{D}(M)+\mathcal{L}(M)+\mathcal{U}(M)$. First we give some lemmas, which will be used in the sequel.

Lemma 2.1. *For any matrix $\widetilde{A}\in C^{n\times n}$ with Schur's decomposition (1.1) we have*

$$\|\Delta\|_F\le\frac{1}{\sqrt{2}}\|\widetilde{A}-\widetilde{A}^*\|_F$$

and

$$\|\Delta\|_F\le\left(\frac{n^3-n}{12}\right)^{\frac{1}{4}}\|\widetilde{A}^*\widetilde{A}-\widetilde{A}\widetilde{A}^*\|_F^{\frac{1}{2}}.$$

Proof. By the decomposition (1.1) we have

$$\begin{aligned}\|\widetilde{A}-\widetilde{A}^*\|_F^2&=\|\widetilde{\Lambda}-\widetilde{\Lambda}^*\|_F^2+\|\Delta-\Delta^*\|_F^2\\&\ge 2\|\Delta\|_F^2,\end{aligned}$$

which implies the first inequality. The second one follows from Theorem 5.3 of [13].

Lemma 2.2. *Let A be a Hermitian matrix, and $\widetilde{A}=A+E$ be a perturbed matrix of A with Schur's decomposition (1.1). Then we have*

$$\|\widetilde{U}^*E\widetilde{U}-\Delta\|_F^2\le\|E\|_F^2+\sqrt{2}\|\Delta\|_F\|E\|_F.$$

Proof. By $E=\widetilde{A}-A$ and the decomposition (1.1) we have

$$\widetilde{U}^*E\widetilde{U}-\Delta=\widetilde{\Lambda}-\widetilde{U}^*A\widetilde{U}. \tag{2.4}$$