Applied Mathematics Letters 65 (2017) 77–82

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

# Sharp estimates for the unique solution of two-point fractional-order boundary value problems

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A R T I C L E I N F O

*Article history:*
Received 13 August 2016
Received in revised form 14 October 2016
Accepted 14 October 2016
Available online 20 October 2016

*Keywords:*
Fractional derivative
Two-point
Boundary conditions
Existence

A B S T R A C T

In this short note, we derive a sharp condition in terms of the end-points of the given interval which ensures the uniqueness of solutions for a Liouville–Caputo type fractional differential equation supplemented with two-point boundary conditions. A comparison with Riemann–Liouville type two-point fractional boundary value problem as well as with the classical two-point problem is presented.
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## 1. Introduction

The idea of fractional derivative dates back to the origin of calculus and was rigorously introduced in the 19th century by Liouville and Riemann. Fractional calculus has been extensively studied and developed during the last few decades. It has been mainly due to an overwhelming interest shown by the modelers and researchers in the subject. One can find the applications of fractional-order derivatives and integrals in several disciplines such as applied mathematics, control theory, mechanical structures and physics. In fact the literature on the topic is now much enriched and covers theoretical development as well as applications of this important branch of mathematics. In particular, the subject of fractional-order boundary value problems has received great attention and a variety of results dealing with the existence and uniqueness theory, and analytic and numerical methods for such problems involving a variety of boundary conditions have been established. For some recent works on the topic, we refer the reader to a series of papers [1–8] and the references cited therein.

In this paper, we consider the following two-point Liouville–Caputo type boundary value problem:

$$\begin{cases} {}^cD^{\beta}x(t) = -f(t,x(t)),\ 1<\beta<2,\ \ t\in[a,b], a,b\in\mathbb{R} \\ \qquad x(a)=\delta_1,\ x(b)=\delta_2,\ \ \delta_i\in\mathbb{R}, i=1,2, \end{cases} \tag{1}$$

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http://dx.doi.org/10.1016/j.aml.2016.10.008
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where ${}^cD^\beta$ denotes the Caputo fractional derivative of order $\beta$, and $f$ is an appropriate continuous functions.

We organize the rest of the contents of the paper as follows. In Section 2, we recall some basic definitions of fractional calculus and prove an auxiliary lemma. Section 3 contains the uniqueness result, while Section 4 presents a detailed discussion.

## 2. Background material

First of all, we describe some preliminary concepts of fractional calculus [9].

**Definition 2.1.** The fractional integral of order $\sigma$ with the lower limit $a$ for a function $\psi$ is defined as

$$I^\sigma \psi(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \frac{\psi(s)}{(t-s)^{1-\sigma}} ds, \quad t > a, \quad r > 0,$$

provided the right-hand side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\sigma) = \int_0^\infty t^{\sigma-1} e^{-t} dt$.

**Definition 2.2.** The Riemann–Liouville fractional derivative of order $\sigma > 0$, $n-1 < \sigma < n$, $n \in N$, is defined as

$$D^\sigma_{a+}\psi(t) = \frac{1}{\Gamma(n-\sigma)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\sigma-1}\psi(s)ds,$$

where the function $\psi$ has absolutely continuous derivative up to order $(n-1)$.

**Definition 2.3.** The Caputo derivative of order $\sigma$ for a function $\psi : [a, \infty) \to R$ can be written as

$${}^cD^\sigma \psi(t) = D^\sigma_{a+} \left( \psi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \psi^{(k)}(a) \right), \quad t > a, \quad n-1 < \sigma < n.$$

**Remark 2.1.** If $\psi(t) \in C^n[a, \infty)$, then

$${}^cD^\sigma \psi(t) = \frac{1}{\Gamma(n-\sigma)} \int_a^t \frac{\psi^{(n)}(s)}{(t-s)^{\sigma+1-n}} ds = I^{n-\sigma}\psi^{(n)}(t), \ t > a, \ n-1 < \sigma < n.$$

Now we prove an auxiliary lemma for the linear variant of problem (1).

**Lemma 2.1.** *A function $x \in C^2[a, b]$ is a solution of problem (1) if and only if it satisfies the integral equation*

$$x(t) = \delta_1 + \frac{t-a}{b-a} \left( \delta_2 - \delta_1 + \frac{1}{\Gamma(\beta)} \int_a^b (b-s)^{\beta-1} f(s, x(s))ds \right) - \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, x(s))ds. \tag{2}$$

**Proof.** As argued in [10], the solution of fractional differential equation in (1) can be written as

$$x(t) = c_1 + c_2(t-a) - \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s, x(s))ds, \tag{3}$$

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