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Properties of a method of fundamental solutions for the parabolic heat equation

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ABSTRACT

We show that a set of fundamental solutions to the parabolic heat equation, with each element in the set corresponding to a point source located on a given surface with the number of source points being dense on this surface, constitute a linearly independent and dense set with respect to the standard inner product of square integrable functions, both on lateral- and time-boundaries. This result leads naturally to a method of numerically approximating solutions to the parabolic heat equation denoted a method of fundamental solutions (MFS). A discussion around convergence of such an approximation is included.

1. Introduction

Meshless methods, in particular the method of fundamental solutions, have gained popularity in recent years both for direct and inverse problems, see the surveys [1] and [2]. It is in particular for stationary problems that research activity on meshless methods has been prolific. For time-dependent problems, typically some transformation in time is used to reduce to the stationary case [3, Section 5]. However, reverting such a transformation can cause numerical problems, see [4, p. 25]. In [5], a method of fundamental solutions for the parabolic heat equation was proposed, and in this method there was no transformation in time. Instead, following the stationary case, linear combinations of the fundamental solution of the heat equation were used. This method has then been applied for various other direct and inverse heat problems, see, for example, [6,7].

A key fact to motivate the MFS in [5] is the linear independence and denseness of linear combinations of fundamental solutions of the heat equation. Proofs thereof in various settings are scattered in those above mentioned works. Therefore, in the present work, we collect the results and shall prove properties of linear independence and denseness for linear combinations of the fundamental solution and derivatives. Moreover, convergence of an MFS approximation will be outlined. This constitutes the novelty of the present work together with pinpointing relevant references for the various results needed in the presented proofs.

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In the present section, we formulate the main result. In Section 2, we collect some results needed in the proof. The proof itself is given in Section 3. In Section 4, we outline a proof of convergence of the MFS approximation. In the final section, Section 5, some remarks are pointed out.

We consider the parabolic heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \ \Omega\times(0,T), \\ \mathcal{B}u = \psi & \text{on } \ \Gamma\times(0,T), \\ u(x,0) = \varphi(x) & \text{for } \ x \in \Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with boundary surface Γ being simple (no self-intersections) closed (the surface has no boundary and is connected) and is at least Lipschitz smooth. The domain Ω_S (S for source) with boundary surface Γ_S, and such that $\overline{\Omega} \subset \Omega_S$, has the similar properties. When $n = 2$, the boundaries of the domains are two simple closed curves with one contained within the other. Doubly-connected domains and also one-dimensional spatial domains can be adjusted for.

The operator $\mathcal{B}u$ denotes either the Dirichlet condition (when $\mathcal{B} = \mathcal{I}$) or the Neumann condition $\mathcal{B}u = \partial u/\partial\nu$, with ν being the outward unit normal to the boundary.

We shall then formulate the main result to be proved. Let

$$F(x,t;y,\tau) = \frac{H(t-\tau)e^{-\frac{|x-y|^2}{4(t-\tau)}}}{(4\pi(t-\tau))^{\frac{n}{2}}} \tag{2}$$

be the standard fundamental solution to the heat equation (1) representing the temperature at location x and time t resulting from an instantaneous release of a unit point source of thermal energy at location y and time τ, with H the Heaviside function. The fundamental solution has the expected physical properties, for example, it is a positive solution to the heat equation for $t > \tau$, for $x \neq y$ there holds $\lim_{t\to\tau^+} F(x,t;y,\tau) = 0$, the function $F(x,t;x,\tau)$ tends to infinity as $t \to \tau^+$, and $\int_{\mathbb{R}^n} F(x,t;y,\tau)\,dy = 1$ when $t > \tau$; for further properties, see [8] and [9, Chapter 1.4–6].

Let $\{y_k, \tau_\ell\}_{k,\ell=1,2,\ldots}$ be a dense set of points on the outer lateral (cylindrical) surface $\Gamma_S\times(0,T)$; notation means that $\{y_k\}_{k=1,2,\ldots}$ is dense on Γ_S and $\{\tau_\ell\}_{\ell=1,2,\ldots}$ is dense in $(0,T)$. By a dense set in L^2, we mean that the span of the set is dense. We can then state the main result:

Theorem 1.1. *The set of functions $\{F(x,t;y_k,\tau_\ell)\}_{k,\ell=1,2,\ldots}$ is linearly independent and dense in $L^2(\Gamma\times(0,T))$. The same hold for the set consisting of the normal derivatives, $\{\partial_{\nu(x)}F(x,t;y_k,\tau_\ell)\}_{k,\ell=1,2,\ldots}$. Moreover, restriction in time generates a set $\{F(x,t_0;y_k,\tau_\ell)\}_{k,\ell=1,2,\ldots}$, which is linearly independent and dense in $L^2(\Omega)$ for any $0 < t_0 \le T$.*

In the next section, we formulate some results needed in the proof. Note that it is for ease of presentation that we chose points on $\Gamma_S\times(0,T)$ as above, that is a dense set in space and a dense set in time. A more general dense set with the points on Γ_S changing with time is also possible.

2. Some results on the parabolic heat equation

The space $L^2(0,T;X)$, where X is a Hilbert space, consists of those measurable functions $u(\cdot,t) : (0,T) \to X$, with $\int_0^T \|u(\cdot,t)\|_X^2\, dt < \infty$. The space $H^k(\Omega)$, $k > 0$, is the standard Sobolev space of functions having weak and square integrable derivatives up to order k, with trace space $H^{k-1/2}(\Gamma)$.

We first recall a well-posedness result for (1):

Proposition 2.1. *Let $\varphi \in L^2(\Omega)$ and let the element ψ be sufficiently regular. Then there exists a unique weak solution $u \in L^2(0,T;H^1(\Omega))$ to (1) with $u_t \in L^2(0,T;L^2(\Omega))$, and this solution depends continuously on the data.*