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Positive solutions of nonlinear delayed differential equations with impulses

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ABSTRACT

The paper is concerned with the long-term behavior of solutions to scalar nonlinear functional delayed differential equations

$$\dot{y}(t) = -f(t, y_t), \ \ t \geq t_0.$$

It is assumed that $f : [t_0, \infty) \times \mathcal{C} \mapsto \mathbb{R}$ is a continuous mapping satisfying a local Lipschitz condition with respect to the second argument and $\mathcal{C} := C([-r, 0], \mathbb{R})$, $r > 0$ is the Banach space of continuous functions. The problem is solved of the existence of positive solutions if the equation is subjected to impulses $y(t_s^+) = b_s y(t_s)$, $s = 1, 2, \ldots$, where $t_0 \leq t_1 < t_2 < \cdots$ and $b_s > 0$, $s = 1, 2, \ldots$. A criterion for the existence of positive solutions on $[t_0 - r, \infty)$ is proved and their upper estimates are given. Relations to previous results are discussed as well.

1. Introduction and the problem considered

Let $C([a, b], \mathbb{R})$ where $a, b \in \mathbb{R}$, $a < b$ be the Banach space of continuous functions mapping $[a, b]$ into $\mathbb{R}$. For $a := -r < 0$ and $b := 0$, we denote this space by $\mathcal{C}$, that is, $\mathcal{C} := C([-r, 0], \mathbb{R})$. Let $\sigma \in \mathbb{R}$, $A \geq 0$, $y \in C([\sigma - r, \sigma + A], \mathbb{R})$. For each $t \in [\sigma, \sigma + A]$, $y_t \in \mathcal{C}$ is defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$.

In the paper, we investigate the long-term behavior of solutions of nonlinear functional delayed differential equations

$$\dot{y}(t) = -f(t, y_t), \ \ t \geq t_0 \tag{1}$$

where $f : [t_0, \infty) \times \mathcal{C} \mapsto \mathbb{R}$, $t_0 \in \mathbb{R}$, is a continuous mapping that satisfies a local Lipschitz condition with respect to the second argument.

Let $\sigma \geq t_0$. In accordance with the traditional definition (we refer, e.g., to [1]), a function $y : [\sigma - r, \sigma + A) \to \mathbb{R}$, where $A > 0$, is called a solution to (1) on $[\sigma - r, \sigma + A)$ if y is continuous on $[\sigma - r, \sigma + A)$, continuously differentiable on $[\sigma, \sigma + A)$ (whenever it is necessary, we will assume that the derivatives in (1) are right-sided),

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satisfies (1) for every $t \in [\sigma, \sigma + A)$ and $(t, y_t) \in [t_0, \infty) \times \mathcal{C}$. For a given $\sigma \in \mathbb{R}$ and $\varphi \in \mathcal{C}$, we say that $y(\sigma, \varphi)$ is a solution of (1) through $(\sigma, \varphi) \in [t_0, \infty) \times \mathcal{C}$ if there is an $A > 0$ such that $y(\sigma, \varphi)$ is a solution of (1) on $[\sigma - r, \sigma + A)$ and $y_\sigma(\sigma, \varphi) \equiv \varphi$. In view of the above conditions, each element $(\sigma, \varphi) \in [t_0, \infty) \times \mathcal{C}$ determines a unique solution $y(\sigma, \varphi)$ of (1) through $(\sigma, \varphi) \in [t_0, \infty) \times \mathcal{C}$ on its maximal interval of existence and $y(\sigma, \varphi)$ depends continuously on initial data [1].

The paper particularly considers the problem of the existence of positive solutions to nonlinear functional delayed differential equation (1) on $[t_0 - r, \infty)$ subjected to impulses

$$y(t_s^+) = b_s y(t_s), \quad s = 1, 2, \ldots, \tag{2}$$

where $t_0 \le t_1 < t_2 < \cdots$, $\lim_{s\to\infty} t_s = \infty$, $y(t_s^+) = \lim_{\varepsilon\to 0, \varepsilon>0} y(t_s + \varepsilon)$ and $b_s > 0$, $s = 1, 2, \ldots$. The number of impulses can be only finite as well. If this is the case, it is sufficient to set $b_s = 1$, $s \ge s_0$ for sufficiently large s_0. Denote $\mathcal{I}_0 = \{t_0, t_1, \ldots\}$, $\mathcal{I} = \{t_1, t_2, \ldots\}$. A solution of the problem (1), (2) is defined as follows. A function $y : [t_0 - r, \infty) \to \mathbb{R}$, is called a solution of the problem (1), (2) on $[t_0 - r, \infty)$ if y is continuous on $[t_0 - r, \infty) \setminus \mathcal{I}$, continuously differentiable on $[t_0 - r, \infty) \setminus \mathcal{I}_0$, satisfies (1) on $[t_0, \infty) \setminus \mathcal{I}_0$, and (2) holds. The main difference of this definition from the classical one (except for taking into account possible discontinuities at points of the set $\mathcal{I}$) is that continuous differentiability rather than continuity is assumed on $[t_0 - r, t_0)$. Later (see Lemma 1), we will prove that such a property is deduced from the method used. A solution y of (1), (2) on $[t_0 - r, \infty)$ is positive if $y(t) > 0$, $t \in [t_0 - r, \infty)$.

The problem of the existence of positive solutions to delayed differential equations is a classical one. For results on the existence of positive solutions to impulsive equations, we refer, e.g., to books [2–4], papers [5–9] and to the references therein. Rudiments of the theory of impulsive equations can be found, in addition to the above books, in [10–12].

The paper is organized as follows. In part 2, a criterion for the existence of positive solutions to the problem (1), (2) on $[t_0 - r, \infty)$ and an upper estimation of such solutions are given. Relations with the previous results are discussed. The proofs of the statements form part 3.

1.1. The idea of approach used

Let us explain the underlying idea of our reasoning. We will look for a positive solution of the problem (1), (2) in the form

$$y(t) = \Lambda(t) := kB(t, t_0) \exp\left(-\int_{t_0 - r}^{t} \lambda(s) ds\right), \quad t \in [t_0 - r, \infty) \tag{3}$$

where $B(t, t_0) := \prod_{t_0 - r \le t_i < t} b_i$ if at least one $t_i \in [t_0 - r, t)$, otherwise we set $B(t, t_0) := 1$, $k > 0$ is a constant and $\lambda : [t_0 - r, \infty) \to \mathbb{R}$ is a continuous function everywhere except for points $t \in \mathcal{I}_0$ where it can have first-type discontinuities. Since $B'(t, t_0) = 0$ if $t \in [t_0 - r, \infty) \setminus \mathcal{I}_0$,

$$y'(t) = -\lambda(t)\Lambda(t), \quad t \in [t_0 - r, \infty) \setminus \mathcal{I}_0. \tag{4}$$

Substitute expressions (3) and (4) for y and y' in (1). Then, the function λ must satisfy an integro-functional equation

$$\lambda(t) = (\Lambda(t))^{-1} f(t, \Lambda_t), \quad t \in [t_0, \infty) \setminus \mathcal{I}_0. \tag{5}$$

In addition, y satisfies (2) since, for $s = 1, 2, \ldots$,

$$y(t_s^+) = \Lambda(t_s^+) = k \left(\prod_{i=1}^{s} b_i\right) \exp\left(-\int_{t_0 - r}^{t_s} \lambda(s) ds\right)$$