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Oscillatory behavior of the second order functional differential equations

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A R T I C L E I N F O

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Establishing new properties of nonoscillatory solutions we introduce new oscillatory criteria for the second order advanced differential equation

$$y''(t) + p(t)y(\sigma(t)) = 0.$$

Our oscillatory results essentially extend the earlier ones. The progress is illustrated via Euler differential equation.

1. Introduction

We consider the second order advanced differential equation

$$y''(t) + p(t)y(\sigma(t)) = 0, \tag{E}$$

where

(H_1) $p(t) \in C([t_0, \infty))$ is positive;
(H_2) $\sigma(t) \in C^1([t_0, \infty))$, $\sigma(t) \geq t$, and $\sigma'(t) \geq 0$.

By a solution of (E) we mean a function $y(t)$ with $y(t)$ in $C^2([t_0, \infty))$, which satisfies Eq. (E) on $[t_0, \infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Eq. (E) is said to be oscillatory if all its solutions are oscillatory.

There are many papers devoted to the oscillation of (E). Various techniques (see [1–18]) have been obtained for investigation of (E), especially for ordinary equations $\sigma(t) = t$ and delay equations $\sigma(t) \leq t$. We mention here the pioneering work of Sturm [1] who introduced comparison principle to the oscillation

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theory. Later Kneser [2] contributes to the subject. A new impetus to the investigation of oscillation was given by Mahfoud [3] who deduces oscillation of delay equation from that of ordinary equation.

The techniques established for ordinary or delay equation in general fail for advanced differential equations. So there are comparatively less oscillatory criteria for advanced equations. Even though Koplatadze et al. [4] presented very nice oscillatory criterion for advanced differential equation, which for the advanced Euler equation

$$y''(t) + \frac{a}{t^2}\, y(\lambda t) = 0, \qquad \lambda \ge 1 \tag{E_1}$$

reduces to

$$a(2 + \ln \lambda) > 1. \tag{1.1}$$

We essentially improve this condition seeing that in our criterion applied for (E_1) the term $\ln \lambda$ will be replaced by power λ^β.

Taking Kusano and Naito's comparison theorem [5] into account, the oscillation of (E) can be deduced from that of ordinary equation

$$y''(t) + p(t)y(t) = 0. \tag{E_2}$$

But this technique does not use informations about value of $\sigma(t)$. Nevertheless, we can establish the first oscillatory criterion.

Theorem A. *Assume that there is a constant a such that for $t \ge t_0$*

$$t^2 p(t) \ge a > \frac{1}{4}, \tag{1.2}$$

then (E) is oscillatory.

Proof. Let us consider the Euler differential equation

$$y''(t) + \frac{a}{t^2} y(t) = 0, \tag{E_E}$$

which is evidently oscillatory for $a > \frac{1}{4}$. Applying Theorem 1 from [5] to Eqs. (E) and (E_E), we conclude that Eq. (E) is oscillatory provided that (1.2) holds. □

In our next considerations we revise (1.2) to contain the advanced argument $\sigma(t)$. In what follows we shall assume that there exists a constant a such that for $t \ge t_0$

$$t^2 p(t) \ge a > 0 \quad \text{and} \quad a \le \frac{1}{4}. \tag{1.3}$$

In the paper, we assume that all functional inequalities hold eventually, that is they are satisfied for all t large enough.

2. Preliminary results

As usually, when studying properties of nonoscillatory solutions of (E), we can restrict our attention only to positive ones.

We recall a well-known lemma of Kiguradze (see [6] or [7]) about the structure of nonoscillatory solutions.