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Hamilton's principle for dynamical elasticity

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ABSTRACT

There exists NO classical variational principle for continuum dynamics so far. The well-known Hamilton's principle is valid only for the conditions prescribed at the beginning and at the end of the motion and is therefore useless to deal with the usual initial-boundary condition problems. In this paper an exact classical variational theory is established, all initial conditions are converted into natural ones, furthermore, all natural final conditions automatically meet the physical requirements, leading to a vital innovation of Hamilton's principle.

1. Introduction

As pointed out by Carni and Genna [1] that the continuum dynamic problem under initial conditions, even in the linear elastic case, does not admit a classical variational formulation, owing to its lack of symmetry with respect to "classical" definitions of bilinear forms [1–3]. Using a modification of Tonti's theory [2,3], Carni and Genna [1] obtained some variational principles with auxiliary variables for the continuum nonlinear dynamics in elasticity.

The only classical variational statement, in continuum dynamics, is Hamilton's principle, which holds only for the conditions prescribed at the beginning and at the end of the motion and is therefore useless to deal with the usual initial condition problems, both as an analytical tool and as a basis for approximate solution methods. It is impossible for most real-life physical problems to prescribe terminal conditions.

The general approach to the establishment of variational formulation for dynamics problems is "non-classical approach", e.g. Gurtin-type variational statement with convolutions, which limits itself to, of course, linear cases.

It would be a landmark in the history of calculus of variations after Hamilton if we can extend Hamilton's principle to all initial-value problems without prescribing both initial and final conditions, see some remarks on Section 5.4.3 entitled as "Modified Hamilton Principles for Initial Value Problems" in Ref. [4] and

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discussion in Ref. [5]. In this paper we will search for such a modification by the semi-inverse method [4–6], which was introduced in the 1997, and has emerged as an interesting and fascinating mathematical tool to the search for various physical problems [7–11].

2. Governing equations

The governing field equations of the continuum dynamic problem, under the assumption of small strains and small displacements, and referred to an orthogonal Cartesian reference frame $x_i, i = 1, 2, 3$, using the Einstein summation convention, can be written as follows [12]

(1) equations of motion:

$$\sigma_{ij,j} + \bar{F}_i = \rho \ddot{u}_i, \quad (\text{in } \Omega \times T) \tag{1}$$

(2) strain–displacement relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (\text{in } \Omega \times T), \tag{2}$$

(3) constitutive law

$$\sigma_{ij} = \frac{\partial A}{\partial \varepsilon_{ij}}, \quad (\text{in } \Omega \times T), \tag{3a}$$

or

$$\varepsilon_{ij} = \frac{\partial B}{\partial \sigma_{ij}}, \quad (\text{in } \Omega \times T). \tag{3b}$$

Here σ_{ij} is the stress tensor, ε_{ij} is the strain tensor, u_i is the displacement vector, $\bar{F}_i$ is the body force, and A and B are strain density and complementary energy density respectively. Ω is the volume of the body, T indicates the considered time interval.

(4) The boundary conditions

On Γ_u the surface displacement is prescribed

$$u_j(x_i, t) = \bar{u}_j(x_i, t), \quad (\text{on } \Gamma_u \times T), \tag{4}$$

and on the complementary part Γ_σ the traction is given

$$\sigma_{ij} n_j = \bar{p}_i, \quad (\text{on } \Gamma_\sigma \times T), \tag{5}$$

where $\Gamma_u \cup \Gamma_\sigma$ covers the total boundary surface.

(5) initial conditions

$$u_j(x_i, 0) = u_i^0(x_i), \quad (\text{in } \Omega), \tag{6}$$

$$\dot{u}_j(x_i, 0) = \dot{u}_i^0(x_i), \quad (\text{in } \Omega). \tag{7}$$

Our aim of this paper is to establish a CLASSICAL variational principle for the above discussed initial-value problem.