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A fast and robust method for computing real roots of nonlinear equations

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ABSTRACT

The root-finding problem of a univariate nonlinear equation is a fundamental and long-studied problem, and has wide applications in mathematics and engineering computation. This paper presents a fast and robust method for computing the simple root of a nonlinear equation within an interval. It turns the root-finding problem of a nonlinear equation into the solution of a set of linear equations, and explicit formulae are also provided to obtain the solution in a progressive manner. The method avoids the computation of derivatives, and achieves the convergence order 2^{n-1} by using n evaluations of the function, which is optimal according to Kung and Traub's conjecture. Comparing with the prevailing Newton's methods, it can ensure the convergence to the simple root within the given interval. Numerical examples show that the performance of the derived method is better than those of the prevailing methods.

1. Introduction

Finding the roots of a nonlinear equation $f(t) = 0$ is a common and important problem in mathematics and engineering computation. Many modified iterative methods have been developed to improve the local order of convergence, including Newton, Halley or Ostrowski's methods. In an iterative step, the convergence order p can be improved by increasing the number n of functional evaluations (FE). The balance between p and n is measured by using $p^{1/n}$, which is called an efficiency index [1–10]. It is conjectured that the order of convergence of any multi-point method cannot exceed the optimal bound 2^{n-1} [11]. Some classical methods, e.g., Newton's method, Newton–Secant method and Ostrowski's method, have efficiency indices of $2^{1/2} \approx 1.414$, $3^{1/3} \approx 1.442$ and $4^{1/3} \approx 1.587$, respectively, and they usually need evaluations of its derivatives. In [12], Li, Mu, Ma and Wang presented a method of sixteen convergence order, whose efficiency index is $16^{1/6} \approx 1.587$. Some optimal methods with eighth convergence order have also been developed, whose

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efficiency index is $8^{1/4} \approx 1.682$ [13–15]. For cases having one root within an interval, the above methods still start from an initial value from one side, and their convergence is sensitive to the selection of the initial value. In some worst cases, the iterative methods cannot converge to the proper simple root, even with a good initial value [16] (see also Example 3 in Section 3 for more details). Chen et al. provided an efficient method based on progressive interpolation, which starts from an interval bounding the unique simple root, and achieves convergence order $3 \cdot 2^{n-3}$ with n FEs and no derivative evaluations [16]. It ensures that the roots of progressive interpolation polynomials are within the given interval, and converge to the simple root.

This paper focuses on how to compute the simple root $t^\star$ of a nonlinear equation within an interval $[a,b]$. It turns the root-finding problem of the given nonlinear equation $f(t)$ into those of a sequence of linear equations. It ensures that each linear equation $l_i(t)$ has a root $t_i \in [a,b]$ with explicit formulae, where t_i is well-approximated to $t^\star$ in a progressive way, $i = 3, 4, \ldots$. It can ensure the convergence to the simple root with convergence order 2^{n-1} by using n FEs without derivative evaluations, which is higher than that of the method in [16] and is optimal in the conjecture of [11] (see also Table 1). Numerical examples show that the performance of the proposed method is better than those of prevailing methods.

2. The main result

Note that a root $t^\star$ of $f(t)$ should be a simple root of $f(t)/f'(t)$. Without loss of generality, we suppose that the simple root $t^\star$ is the unique root of $f(t)$ within $[a,b]$, and $h = b - a$. In cases with only one given initial value x_0, one can obtain x_1 such that $|x_1 - t^\star| < |x_0 - x^\star|/2$ by using some iterative methods, such as the Newton's method, and can thus obtain an interval bounding $t^\star$. The details are as follows. If $f(x_0) \cdot f(x_1) < 0$, x_0 and x_1 already bound the simple root $t^\star$; otherwise, we have that $2x_1 - x_0$ and x_1 bound $t^\star$.

In this paper, we discuss how to compute the simple root $t^\star \in [a,b]$ of $f(t)$, where $f(a) \cdot f(b) < 0$. We want to find a sequence of linear equations which have roots well-approximating to $t^\star$. Let $t_1 = a$ and $t_2 = b$. Let the first linear equation $l_3(t)$ be the one interpolating $f(t)$ at two points t_1 and t_2, which is trivial to compute its root t_3. Let

$$l_i(t) = \frac{t - t_i}{\sum_{j=2}^{i-1} \alpha_{i,j} t^{j-2}}, i = 3, 4, \ldots, n, \tag{1}$$

where the $i-1$ unknowns, i.e., t_i and $\alpha_{i,j}$, $j = 2, 3, \ldots, i-1$, are determined by

$$l_i(t_j) = f(t_j), j = 1, 2, \ldots, i-1. \tag{2}$$

Note that the denominator of $l_i(t)$ should not be zero within $[a,b]$, and $l_i(t) = 0$ is equivalent to a linear equation with root t_i. On the other hand, combining the assumption that $f(a) \cdot f(b) < 0$ and the constraints that $l_i(a) = f(a)$ and $l_i(b) = f(b)$, $l_i(t)$ must have the root t_i. We introduce Theorem 3.5.1 in Page 67, Chapter 3.5 of [17] as follows.

Theorem 1. *Let $w_0, w_1, \ldots, w_r$ be $r+1$ distinct points in $[a,b]$, and $n_0, \ldots, n_r$ be $r+1$ integers ≥ 0. Let $N = n_0 + \cdots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that*

$$g^{(i)}(w_j) = f^{(i)}(w_j), \quad i = 0, \ldots, n_j, \quad j = 0, \ldots, r.$$

Then there exists $\xi_1(t) \in [a,b]$ such that

$$f(t) - g(t) = \frac{f^{(N+1)}(\xi_1(t))}{(N+1)!} \prod_{i=0}^{r} (t - w_i)^{n_i}. \quad \square$$