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# A new numerical method for variable order fractional functional differential equations

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A B S T R A C T

In this letter, a high order numerical scheme is proposed for solving variable order fractional functional differential equations. Firstly, the problem is approximated by an integer order functional differential equation. The integer order differential equation is then solved by the reproducing kernel method. Numerical examples are given to demonstrate the theoretical analysis and verify the efficiency of the proposed method.



## 1. Introduction

Fractional differential equations have been applied successfully in earthquake analysis, electric circuits, bio-chemical, controller design, signal processing, etc. Therefore, fractional differential equations have attracted much attention. Recently, the concept of variable order operator has been introduced in [1–4]. An increasing number of problems in mathematical physics and engineering have been modeled via variable order fractional differential equations [1–4].

Due to the existence of variable fractional derivative, it is usually impossible to obtain the analytical solution of such equations. Hence, it is important to develop numerical methods for solving such problems. Cao and Qiu [5] derived a high order numerical scheme for variable order fractional ordinary differential equation by establishing a second order numerical approximation to variable order Riemann–Liouville fractional derivative. Fu, Chen and Ling [6] applied the method of approximate particular solutions to both constant- and variable-order time fractional diffusion models. Liu, Shen, Zhang, et al. [7–12] proposed various finite difference methods for variable order fractional partial diffusion equations. Sierociuk, Malesza and

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Macias [13] introduced a numerical scheme for a variable order derivatives based on matrix approach. Yu and Ertürk [14] applied a finite difference method to variable order fractional integro-differential equations. Zhao, Sun and Karniadakis [15] derived two second-order approximation formulas for the variable-order fractional time derivatives. Zayernouri and Karniadakis [16] developed fractional spectral collocation methods for linear and nonlinear variable order fractional partial differential equations. Combining Legendre wavelets functions and operational matrices, Chen, Wei, et al. [17] presented a numerical method to solve a class of nonlinear variable order fractional differential equations. Atangana [18] gave the Crank–Nicholson scheme for time-fractional variable order telegraph equation. Bhrawy and Zaky [19] provided the numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. Wand and Vong [20] proposed compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation. Li and Wu [21,22] proposed a reproducing kernel method for variable fractional boundary value problems.

Based on the reproducing kernel theory, the reproducing kernel method proposed by Cui and Geng, et al. [23–25] has been developed and applied to many fields [26–41]. In this letter, we consider the following variable order fractional functional differential equation

$$\begin{cases} D^{\alpha(t)}u + a(t)u'(t) + b(t)u(\tau(t)) = f(t),\ 0 < t < 1, \\ u(0) = \mu_0, \end{cases} \tag{1.1}$$

where $0 < \alpha(t) \leq 1$, $0 \leq \tau(t) \leq 1$, $\mu_0$ is a constant, $a(t)$, $b(t)$ and $\tau(t)$ are sufficiently smooth, $D^{\alpha(t)}$ denotes the variable order Caputo fractional derivative defined as follows

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-s)^{-\alpha(t)}u'(s)ds. \tag{1.2}$$

## 2. Numerical method for Eq. (1.1)

If $u(s) \in C^n[0,1]$, we can construct the n$th$ Taylor polynomial for $u'(s)$

$$P_n(s) = u'(t) + u''(t)(s-t) + \frac{u'''(t)}{2!}(s-t)^2 + \cdots + \frac{u^{(n)}(t)}{(n-1)!}(s-t)^{n-1}.$$

Because $P_n(s)$ is a polynomial, we can exactly determine the value of

$$\int_0^t (t-s)^{-\alpha(t)}P_n(s)ds = \sum_{k=0}^{n-1}\frac{(-1)^k t^{k+1-\alpha(t)}}{k!(k+1-\alpha(t))}u^{(k+1)}(t). \tag{2.1}$$

When the Taylor polynomial $P_n(s)$ agrees closely with $u'(s)$ throughout the interval $[0,1]$, (2.1) is generally the dominant portion of the approximation of $\int_0^t (t-s)^{-\alpha(t)}u'(s)ds$.

From (1.1), we can furthermore get some initial conditions

$$u'(0) = \mu_1,\ \ u''(0) = \mu_2, \ldots, u^{(n-1)}(0) = \mu_{n-1}.$$

Hence, (1.1) can be approximated by an integer order differential equation

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha(t))}\displaystyle\sum_{k=0}^{n-1}\frac{(-1)^k t^{k+1-\alpha(t)}}{k!(k+1-\alpha(t))}u^{(k+1)}(t) + a(x)u'(t) + b(t)u(\tau(t)) = f(t),\ 0 < t < 1, \\ u(0) = \mu_0,\ u'(0) = \mu_1,\ \ u''(0) = \mu_2, \ldots, u^{(n-1)}(0) = \mu_{n-1}. \end{cases} \tag{2.2}$$

We will solve high order differential equation (2.2) by a direct way rather than by transforming it to a system of first order differential equations. Following we introduce the reproducing kernel method proposed for Eq. (2.2).

First, we construct reproducing kernel spaces $W^{n+1}[0,1]$, $(n \geq 2)$ in which every function satisfies the initial conditions of (2.2).